ARBITRARY MOTION OF AN ELONGATED BODY IN AN IDEAL FLUID

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A method is considered which permits the use of a computer to calculate the fluid velocity on the surface of a solid body moving in an ideal fluid and to calculate the added mass. The method of [1, 2], developed for bodies of revolution, in which the flow is simulated by a system of sources and sinks distributed continuously over the body surface, is extended to the case of an arbitrary body. In contrast with the analogous work of Hess and Smith [3], where the fluid velocities on the surface of an arbitrary body were determined for translational motions, in the present case the basic integral equation of the problem is solved by the method of successive approximations without preliminary approximation of the equation by a system of linear algebraic equations of high order, which leads to a shortening of the computations.

The results of the calculations are compared with the known exact values of the velocities and the added mass for a triaxial ellipsoid, and also with the results of the experimental determination of the pressures on the surface of an elongated body.

1. Basic relations. If $V_1$, $V_2$, $V_3$ are the projections of the translational velocity vector $V$ of the pole $A$ of a solid body, and $\Omega_1, \Omega_2, \Omega_3$ are the projections of the angular velocity vector $\Omega$ on the $xyz$ coordinate system fixed in the body (Fig. 1), then we can write for the disturbed velocity potential of the fluid

$$
\Phi(x, y, z, t) = \sum_{i=1}^{6} V_i(t) \Phi_i(x, y, z).
$$

Here the notation is ($l$ is the body length)

$$
V_1 = l \Omega_1, \quad V_2 = l \Omega_2, \quad V_3 = l \Omega_3.
$$

The external Neumann problem must be solved in order to determine each of the six unit potentials

$$
\Phi_i(x, y, z),
$$

with zero conditions at infinity. Here $V_i$ are the velocities of the points of the body surface in the corresponding simple motion, $S$ is the surface bounding the solid body, $n$ is the normal to the surface $S$, directed within the fluid. If the solution is sought in the form of the potential of a simple layer

$$
\Phi_i(P) = -\int_{S} \mu_i(\Omega) \frac{dS}{\Omega_i},
$$

then the boundary condition (1.1) reduces to the Fredholm integral equation of second kind relative to the layer intensity $\mu$,

$$
2\mu_i(P) + \int_{S} \mu_i(\Omega) \frac{R \cdot n(P)}{R} dS = V_i(P) \cdot n(P),
$$

where $P$ is an arbitrary reference point, $Q$ is any point of the surface $S$, the vector $R = QP$. Equation (1.3) has a unique solution if the surface $S$ belongs to the Lyapunov class [4].

The longitudinal $x$-axis of the Cartesian coordinate system $xyz$ with the unit vectors $i, j, k$ is drawn between the most distant points of the body. The coordinate origin is located at the nose of the body. Along with the Cartesian system, we consider the cylindrical coordinate system $x, r, \theta$ ($y = r \cos \theta, z = r \sin \theta$). The coordinates $x, r, \theta$ are assigned to the reference point $P$, the coordinates $\xi, \rho, \phi$ are assigned to the variable point $Q$. The vector

$$
R = (x - \xi)i + (r \cos \theta - \rho \cos \phi)j + (r \sin \theta - \rho \sin \phi)k.
$$

If the pole $A$ is selected on the $x$-axis at the distance $x_A$ from the nose of the body, then we can write for the velocity of the points of the body surface for arbitrary body motion

$$
V = [V_x r \sin \theta - V_x r \cos \theta - V_x l]i + [V_x (x - x_A) - V_x r \sin \theta - V_x l]j + [V_x r \cos \theta - V_x (x - x_A) - V_x l]k,
$$

where the linear dimensions are referred to the body length $l$, and the positive directions of the components of the translational velocity are considered to be the directions opposite to the directions of the $x, y, z$ axes (Fig. 1).

The body surface is specified in the form of the relation

$$
r = r(x, \theta).
$$

An element of the surface $S$ is equal to

$$
dS = r \sqrt{1 + \rho^2 + \phi^2} d\theta d\phi, \quad p = \frac{\partial r}{\partial \theta}, \quad q = \frac{1}{r} \frac{\partial r}{\partial \phi}.
$$

At each point of the body surface we introduce a system of rectangular coordinates fixed with this point, whose unit vectors are

$$
n = [1 + p^2 + \phi^2]^{-1/2} [-pi + (q \cos \theta - \cos \phi)j - (q \cos \theta - \sin \phi)k],$$

$$
\tau = [(1 + p^2 + \phi^2)(1 + q^2)]^{-1/2} \times [\phi(q \sin \theta + \cos \phi)j + p(q \cos \theta - \sin \phi)k],$$

$$
b = [1 + q^2]^{-1/2} [q(q \cos \theta - \sin \phi)j + (q \sin \theta + \cos \phi)k].$$

Here the vector $n$ is normal, the vector $b$ coincides with the line of intersection of the tangent plane and the transverse plane $x = \text{const}$, the vector $\tau$ has a positive and nonzero projection on the longitudinal $x$-axis, in connection with which the velocity projections on the $\tau$ axis may be termed longitudinal, those on the $b$ axis are termed transverse.

In connection with the selected form (1.4) and (1.5) for representing the surface, limitations are imposed on the surface. Thus, there must not be any planes tangent to the surface which are perpendicular to the $x$-axis (p bounded), other than the end points, and there must not be any tangent planes passing through the x-
axis (qr bounded). At the end points, where r → 0 and qr is limited, p → ∞ is admitted under the condition of boundedness of the limit of the product pr in any meridional section θ = const. Geometrically this means that the planes drawn through the end points perpendicular to the x axis have only a single point of tangency with the body surface—the end point. Generally speaking, the surface of the majority of the bodies used satisfies this requirement.

In place of the intensity μi of the simple layer in each simple motion we introduce the new functions gi, connected with μi by the relations

$$g_i = 2πr^\frac{1}{2} + p^2 + q^2 u_i / V_i \quad (i = 1, 2, \ldots, 6).$$

After elementary transformations the integral equation (1.3) for each simple motion reduces to the form

$$g_i(x, 0) = f_{i0}(x, 0) - \int_0^{2\pi} g_i(ξ, θ)K_0(x, θ, 0, θ) dξ dθ, \quad (1.6)$$

where K_0 is the equation kernel, which is independent of the type of motion

$$K_0 = \frac{r [r - p (x - ξ) - p (cos (θ - θ)) - qr sin (θ - θ)]}{2π [(x - ξ)^2 + r^2 + p^2 - 2rp cos (θ - θ)]^{1/2}},$$

and for the known functions f_{i0} (x, θ) we have introduced the notation

$$f_{i1} = pr, \quad f_{i2} = r (q cos θ + sin θ), \quad f_{i3} = -q r^2, \quad f_{i4} = r (q cos θ + sin θ),$$

$$f_{i5} = pr, \quad f_{i6} = -q (sin θ - q cos θ),$$

$$f_{i7} = -q (cos θ - cos θ), \quad f_{i8} = -r,$$

where C_{i0} = 1, the remaining C_{i1} (i = 2, 3, \ldots, 6) are equal to zero. For x = 0 we need only replace unity by zero under the integral sign in (1.8).

For the unit disturbed velocity potentials we obtain the equations

$$\Phi_i(x, θ) = -\int_0^{2\pi} g_i(ξ, θ) dξ dθ \quad (1.10)$$

For the end points of the body the equations for the relative velocities and unit potentials are simplified similarly to (1.8).

The known expressions for the added mass coefficients

$$\lambda_{ij} = -ρ_0 \int_0^l f_0(x, θ) \Phi_j(x, θ) dx dθ \quad (i, j = 1, 2, \ldots, 6), \quad (1.11)$$

where ρ_0 is the fluid mass density, f_{i0} (x, θ) are determined by (1.7).

2. Computation methods. The determination of the solution g_i(x, θ) of the integral equations (1.3) is accomplished at a finite number of discrete reference points of the body surface. Thus, in compiling the machine program to be used in computing the examples, we provided 24 reference transverse sections x_i = const at arbitrary intervals of the x-axis on half of the body which had the plane of symmetry x0y, and 13 reference meridional sections θ_j = const were provided at arbitrary intervals of the angle θ, beginning at θ = 0 and ending at θ = 180°. The reference points were at the intersections of these transverse and meridional sections on the body surface. Thus, the number of reference points in the examples considered, including the two end points, was 314.