THREE-DIMENSIONAL FULLY ESTABLISHED FLOW
OF AN IDEAL LIQUID AROUND THE BLADE
SYSTEMS OF TURBOMACHINES

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The present-day methods for the quasi-three-dimensional calculation of flow around the working organs of turbomachines were developed in [1, 2]. The most widely used in practice is the method of [2] (see, for example, [3, 4]), which, in a complete statement, proposes the solution of three known two-dimensional problems at the coordinate surfaces of a triorthogonal natural system of coordinate surfaces in the flow-through part of a turbomachine. A method is proposed below for the hydrodynamic calculation of blade systems directly in the three-dimensional region, based on solutions of the integral equations of the three-dimensional theory of a field, and not connected with any kind of metric schematization of the flow of the liquid. Comparative analyses show that, with the use of an electronic computer, the present method and the method of [2] (in the complete statement) are practically equivalent.

§ 1. The steady-state flow of an ideal barotropic liquid in a closed region $\mathbf{V}$ with a piecewise-smooth boundary $S$, in the presence of a conservative field of volumetric forces with the potential $U$, can be described by the system of equations

\begin{align}
\nabla \times \mathbf{v} &= \mathbf{\Omega}, \quad \nabla \mathbf{v} = \mathbf{R} \\
\frac{d\mathbf{\Omega}}{dt} &= (\nabla \mathbf{v}) \mathbf{v} - \mathbf{\Omega} \mathbf{R} \\
\nabla F(p) &= -\mathbf{\Omega} \times \mathbf{v} - \nabla \left( U + \frac{v^2}{2} \right), \quad F(p) = \int \frac{dp}{\rho_0}, \quad v = |\mathbf{v}| \\
\rho_0 &= \rho_s(p) \\
R &= -\frac{1}{\rho_s}(\mathbf{v} \nabla \rho_s)
\end{align}

containing five sought functions of the field: the vectors of the velocity $\mathbf{v}$ and the curl $\mathbf{\Omega}$, as well as the pressure $p$, the density $\rho_0$, and the divergence $R$. In (1.1)-(1.5) the symbol $\nabla$ is used for the Hamiltonian operator.

To the system of equations (1.1)-(1.5) there must be added the boundary conditions

\begin{align}
\mathbf{v}_n = \mathbf{v}_0(Q \in S), \quad \int_S \mathbf{v}_n(Q) dS = \int_v R(P) dV; \quad \mathbf{\Omega}(Q \in S_\bot) = \mathbf{\Omega}_\bot
\end{align}

where $\mathbf{v}_n(Q)$ is the normal component of $\mathbf{v}$ in $S$; $S_\bot \subset S$ is a part of the boundary $V$, intersecting all the lines of flow once. It is assumed that $\mathbf{\Omega}_\bot$ and $\mathbf{v}_n$ are continuous at $S$ and at smooth elements of $S$, respectively. The velocities in the region $V$ are assumed to be subsonic.

From the structure of the system (1.1)-(1.5) it follows that the proposed statement of the problem of the complete description of the flow of an ideal liquid can be implemented by the method of successive approximations in accordance with the following scheme. In each approximation, from Eqs. (1.1), in accordance with correctly assigned (on the basis of the preceding approximations) functions $\mathbf{\Omega}$, $R$, the field of the vector $\mathbf{v}$ is
determined, corresponding to the first boundary condition of (1.6). We shall assume that the functions $\Omega$ and $R$ are correctly assigned if they are continuous and they satisfy the necessary conditions for solvability of the Eqs. (1.1) [5].

Then the general solution of Eqs. (1.1) is written in the form [5]

$$v(P_0) = \frac{1}{4\pi} \int \left[ v(Q) (\nabla r^{-1}(P_0, Q) n(Q)) + v(Q) \times (\nabla r^{-1}(P_0, Q) \times n(Q)) \right] dS + \frac{1}{4\pi} \int [R(P) \nabla r^{-1}(P_0, P) + f(P) \times \nabla r^{-1}(P_0, P)] dV$$

(1.7)

The following notation is adopted: $r$ is the distance between fixed $P_0 \in V$ (or $Q_0 \in S$) and instantaneous $P \in V$ (or $Q \in S$) points; $n$ is the unit vector of the normal to $S$. The values of the velocities $v(Q)$ in the right-hand part of Eq. (1.7) are determined by solution of a system of singular integral equations, obtained from (1.7) by passing to the limit with $P_0 \to Q_0$ [5].

Then the Helmholtz equation (1.2) is solved. As $S$ we take the surface through which the liquid flows into the volume under consideration $V$. Taking into consideration now that the substantive derivative is taken with respect to the time $t$, and carrying out the integration along the trajectories of the liquid particles, we obtain

$$\Omega(s) - \Omega(s_0) = \int_{s_0}^{s} \left[ (\Omega(\sigma) v(\sigma) - \Omega(\sigma) R(\sigma) \right] d\sigma$$

or

$$\Omega(s) = \Omega(s_0) + \int_{s_0}^{s} \frac{1}{v(\sigma)} \left[ (\Omega(\sigma) v(\sigma) - \Omega(\sigma) R(\sigma) \right] d\sigma$$

where $\sigma$ and $s$ are the instantaneous and final points of the integration along the trajectory of each fixed liquid particle.

Taking as $t_0$ the moment of time at which a particle is located at the point $s_0 \in S_\infty$, we finally obtain

$$\Omega(s) = \Omega_\infty + \int_{s_0}^{s} T_{kl}(\sigma) \Omega(\sigma) d\sigma, \quad T_{kl} = \frac{1}{v} \left( \frac{\partial v_k}{\partial x_l} - \delta_{kl} R \right)$$

(1.8)

Here $T_{kl}$ is an orthogonal tensor of the second rank; $(x_i)$ is a Cartesian system of coordinates; $\delta_{kl}$ is a Kronecker symbol; $v_k$ are the components of $v$; $k, l = 1, 2, 3$. Equality (1.8) is a partial case of an integral Volterra equation of the second kind.

Limiting ourselves in (1.8) to trajectories for which $\sigma \in \varepsilon^*$, where $\varepsilon^*$ are small neighborhoods of the critical points $v = 0$, and taking account of the continuity of all the first derivatives of $v$ (by virtue of the conditions adopted above with respect to the functions in the right-hand parts of (1.1), (1.6), and the properties of the so-