Scaling Factors Associated with M-Furcations of the $1 - \mu |x|^z$ Map

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A numerical study is made of the scaling behavior associated with M-furcations ($M = 3, 4, 5$) in the map $x_{t+1} = 1 - \mu |x_t|^z$ ($z > 1$). The scaling constants $\delta$ and $\alpha$ are calculated as functions of $z$, as well as the more general scaling functions $\sigma$ and $f(\alpha)$.

KEY WORDS: Chaos; one-dimensional maps; multifractality.

1. INTRODUCTION

The one-dimensional iterative map

$$x_{t+1} = f(x_t) = 1 - \mu |x_t|^z, \quad z > 1$$

which maps the interval $x \in [-1, 1]$ into itself, displays a very rich dynamical behavior. This map is generic for all single-hump one-dimensional maps which have (locally around the maximum) a leading nonlinearity of order $z$. The $z=2$ case is by far the most common in experiments, but other values of $z$ are also found.

When the parameter $\mu$ in Eq. (1) is raised (starting from $\mu = 0$) the attractors (or long-time solutions) of the map show a sequence of periodic orbits with period $2^k$ ($k = 0, 1, 2...$). The $k$th period appears at $\mu_k$ through a pitchfork bifurcation of the $(k-1)$th period, and the sequence $\{\mu_k\}$ accumulates ($k \to \infty$) at $\mu_\infty(z)$, where the system enters into chaos. In the chaotic region aperiodic attractors are present as well as an infinite number of periodic windows, which always appear in the same order, independently of $z$. When these windows are taken in an appropriate order, they

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form sequences of $M$-furcations, with period $M^k$ ($M > 2$), which generalize the bifurcations ($M = 2$). The windows of the $M$-furcations are not adjacent on the parameter axis and they are very narrow. However, the period-tripling sequence ($M = 3$) has been observed experimentally.\(^{(5)}\) Above $\mu = 2$ no finite attractors exist and $x_t$ is driven to infinity.

All the sequences of $M$-furcations present scaling factors that converge and define universality classes determined by $z$, in the sense that the factors do not change if higher-order terms are included in Eq. (1). In the $\mu$ direction there is the scaling factor $\delta$ and in the $x$ direction there is a whole set of scaling factors (the principal ones being $\alpha$ and $\alpha^2$) which together form the scaling function $\sigma$. The existence of a set of scaling indices in the attractor at the accumulation point of the $M$-furcation characterizes the presence of a multifractal, which can be studied through the function $f(a)$.

Van der Weele \textit{et al.}\(^{(6)}\) studied $\delta$, $\alpha$, $\sigma$, and $f(a)$ as function of $z$ for the bifurcations ($M = 2$), and Shau-Jin Chang \textit{et al.}\(^{(7)}\) calculated $\alpha$, $\delta$, and fractal dimensions for the $z = 2$ case and $M \leq 7$. References 8–14 also deal with scaling factors for the $M$-furcations in the map (1).

The aim of the present communication is to study numerically, as function of $z$, the scaling factors $\alpha$ and $\delta$, and the scaling functions $\sigma$ and $f(a)$ for $M = 3, 4, \text{ and } 5$, which correspond, respectively, to trifurcations, tetrafurcations, and pentafurcations. The paper is organized as follows: in the next section I calculate the scaling function $\delta$; Sections II and III study the functions $\sigma$ and $f(a)$, respectively; the last section is dedicated to the conclusions.

2. THE SCALING FACTOR $\delta$

In this section let us initially fix upon notations before introducing the method used in the numerical calculations. For every periodic orbit in the map (1) there is one value of the control parameter for which the orbit includes the critical point (peak) of the map. At this value of the parameter the cycle is called superstable. Following the images of the peak at the superstable cycle it is possible to form a word of $R$ and $L$ according to whether the subsequent iterations in the orbit are on the right or on the left of the peak. This word is called the U-sequence of the cycle.\(^{(15)}\) In the case of the trifurcations and tetrafurcations the basic 3-cycle and 4-cycle have U-sequences $RL$ and $RLL$ (or $RL^2$). The pentafurcations have three types of sequences for the basic 5-cycle, namely $RLR^2$, $RL^2R$, and $RL^3$. The U-sequences related to the higher-order periods in the $M$-furcations are constructed following the rules described in refs. 1 and 15.

For each family of cycles related to a sequence of $M$-furcations the set