Because of the first of the expressions (27),

\[ 2\psi_{r}^{-}\left(0\right) = -\psi_{r}^{-}\left(0\right). \tag{52} \]

We substitute (52) and (53) in (36) and (44). Bearing in mind that by virtue of the spectral condition \( \tilde{\psi}_{r}^{-}\left(\xi\right) = 0, \ \xi > 1, \) and \( \psi_{r}^{-}\left(r\right) = \psi_{r}^{-}\left(-r\right), \) we obtain the required proposition.

Proposition C is a consequence of Eqs. (58)–(40) and (45).

Remark 1. It follows from Proposition A that at least one of the functions \( F_{s}^{(5)}, \ F_{x}^{(5)} \) has a purely scaling behavior in the Bjorken limit (provided these functions do not increase as \( \nu \to \infty \)).

Remark 2. To derive the sum rule (49), we assumed the validity of the Callan–Gross relation (48) from which it followed that \( c_{s}, \psi_{r}^{-}\left(r\right) = 0. \)

Obviously, for the derivation of (49) the weaker assumption \( \psi_{r}^{-}\left(0\right) = 0 \) is sufficient. However, it is not clear what is the physical meaning of such an assumption. We note that in Bjorken's paper [5] the sum rule (49) was obtained from entirely different arguments (using dispersion relations, the scaling hypothesis, and the equal-time commutation relations between the currents).

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LITERATURE CITED


ASYMPTOTIC SOLITON-LIKE SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION WITH VARIABLE COEFFICIENTS

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Asymptotic soliton-like solutions of the nonlinear Schrödinger equation are constructed. Necessary and sufficient conditions for the existence of such solutions are found.

We consider the nonlinear Schrödinger equation with variable coefficients

\[ i\varepsilon q_{x}(x, t) + \varepsilon^{2}q_{tt}(x, t) + \nu(x, t)|q(x, t)|^{2}q(x, t) = 0, \tag{1} \]

which has numerous applications in physics. Here, \( x \in \mathbb{R}, \ t \in \mathbb{R}, \ q(x, t) \in \mathbb{C}^{\infty} \) is an unknown complex-valued function, \( \nu(x, t) \in \mathbb{C}^{\infty} \) is a real function, \( \nu < 0, \ \varepsilon \in (0, 1) \) is a small parameter. Subscripts of the unknown \( q \) denote differentiation with respect to the corresponding variables.

Equation (1) with \( \nu = \text{const} \) has been well studied (see, for example, the bibliography in [1]); with variable \( \nu, \) less well. In [2] Grimshaw constructed an asymptotic solution whose leading term has the form of a soliton solution with time-dependent parameters. In the present paper, we construct solutions of Eq. (1) in the asymptotic limit \( \varepsilon \to 0 \) with variable \( \nu(x, t) < 0. \)

Suppose that in Eq. (1) \( \nu = -1. \) Then this equation has solutions in the form of a solitary wave propagating through an oscillating background:

\[ q(x, t, \varepsilon) = \exp\left\{ \left(\omega_{1} + \kappa_{-}^{2} \arctg [x_{1} \text{th}(\omega_{1}/\varepsilon)]\right)\left[x_{2} - \beta \cdot \text{ch}^{-1}(\omega_{2}/\varepsilon)\right]\right\}, \tag{2} \]

where \( \omega_{1} = x - 2t, \ \omega_{2} = (x - t)/2, \ \alpha = 1, \ \beta = 1/2, \ \kappa_{-} = -1/4, \ k_{2} = 1. \) A natural problem which arises in the study of Eq. (1) with variable \( \nu = \nu(x, t) < 0 \) is the following: Does Eq. (1) have solutions analogous to (2)? We show that
such (asymptotic) solutions exist and give corresponding expressions. These solutions represent a "distorted" soliton propagating through a rapidly oscillating background. The expression for the leading term of this solution has the form (2), but the parameters $\alpha, \beta, \kappa, \omega, \omega_1$ in (2) become smooth functions of $x$ and $t$. We shall say that such solutions are soliton-like solutions on a rapidly varying background, or, for brevity, soliton-like solutions (we recall [9] that a function $q_n = O(1)$ whose substitution in (1) gives $O(\varepsilon^n)$, $n \geq 0$, solution of Eq. (1). A function $q_0 = O(1)$ such that $q_n - q_0 = O(\varepsilon^n), \sigma > 0$, is called the leading term of the asymptotic solution. The symbol $O(\varepsilon^n)$ denotes a function $X(x, t, \varepsilon)$ that on any compactum $K \subset \mathbb{R}^2$ satisfies the estimate $|X| \leq C_n \varepsilon^n$, where $C_n \kappa$ is a constant which depends on $n$ and $K$). We formulate the main result, beginning with the introduction of some auxiliary functions.

Suppose that for $t \in [0, T]$ the real functions $\Phi(x, t), \kappa(x, t) \in C^\infty$ satisfy the system of equations

$$
\Phi_t + \Phi_\kappa \kappa = 0, \quad \kappa_t + 2\Phi_x \kappa + \Phi_\kappa \kappa x = 0. \tag{2}
$$

Note that (3) is a system of nonlinear differential equations of hyperbolic type whose characteristic velocities are $c^\pm = \pm \frac{1}{2\sqrt{2\nu^2}}$. In (3) we set $2\Phi_x = u, \kappa^2 = \rho$. It is readily seen that the system (3) is analogous to the system of "shallow-water" equations

$$
u_t + uu\kappa = - \frac{1}{\theta} \rho_t, \quad \rho_t + (\rho u) = 0
$$

with equation of state in the form $dp = -2\rho d(\nu \kappa)$, where $u$ is the velocity of the liquid, $\rho$ is the density, and $p$ is the pressure.

Suppose that for $t \in [0, T]$ the real function $\psi = \psi(t) \in C^\infty$ satisfies the ordinary differential equation

$$
a = a(x, t) = (2\Phi_x - \psi); \quad \frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \psi \frac{\partial}{\partial x} \text{ is the derivative along the trajectories } dx/dt = \dot{\psi}, \quad 0 < |\alpha| < \frac{1}{2\sqrt{2\nu^2}}.
$$

Suppose that on the same $t$ interval the real function $\rho_0 = \rho_0(t) \in C^\infty$ satisfies the ordinary differential equation

$$
L \rho_0(t) = \left( - \frac{d^2}{dt^2} + b(\psi(t), \psi, \tau)b(\psi_\psi, \psi_\tau, t) \right) \rho_0(t) = \mathcal{H}_0(t), \tag{6}
$$

where $a = a(x, t) = (2\Phi_x - \psi)$ is the relative velocity of motion of some point $x = \psi(t)$, $d/\partial t = \partial/\partial t + \psi \partial/\partial x$ is the derivative along the trajectories $dx/dt = \dot{\psi}$, and

$$
0 < |\alpha| < \frac{1}{2\sqrt{2\nu^2}}.
$$

And suppose the real functions $\mu^+(x, t), \mu^-(x, t) \in C^\infty$ for $t \in [0, T]$ satisfy the system of equations

$$
\mu_t + 2\Phi_\mu \mu^+ + 2\Phi_x \mu^+ + (g^+ \mu^+) = 0, \quad g_t + 2\Phi_x g^+ - 2\nu g^+ = 0,
$$

and the boundary conditions on the curve $x = \psi(t)$

$$
\left. \left( \frac{\mu^+ - \mu^-}{a} - \frac{1}{a} \frac{d}{dt} \left( \frac{A}{a} \right) \right) \right|_{x = \psi(t)} = 0, \quad \left( \frac{\mu^+ - \mu^-}{a} - \frac{1}{a} \frac{d}{dt} \left( \frac{A}{a} \right) \right) \left|_{x = \psi(t)} = 0, \tag{8}
$$

where $A = A(x, t) = \frac{1}{2\sqrt{2\nu^2}} - a^2$. We introduce the notation

$$
f(t, x, t) = \frac{1}{2\sqrt{2\nu^2}} - \frac{1}{a} \frac{d}{dt} \left( \frac{A}{a} \right), \quad \gamma(t, x, t) = \frac{1}{2} \left[ \left( g^+ (x, t) + g^- (x, t) \right) + \left( g^+ (x, t) - g^- (x, t) \right) \right] \theta(t, x, t).
$$

The functions $f, \theta, \gamma \in C^\infty$ are real. We introduce the function

$$
q_0 = e^{i\Phi(x, t)/\theta} f(t, \tau, \rho_0(t), x, t) e^{i\psi(t) / \rho_0(t)}, x, t \big|_{x = \psi(t)}.
$$

THEOREM. Suppose that on the interval $t \in [0, T]$ the functions $\Phi, \kappa, \psi, \rho_0, \mu^+, \mu^-, f, \theta, \gamma, q_0$ satisfy the conditions (3)-(10). Then the function $q_0$ is the leading term of the asymptotic solution of Eq. (1).