Frequency of Quasi-Geostrophic Modes on Hexagonal Grids

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With 6 Figures

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Summary

Finite-difference analysis of Rossby modes has been performed for two staggered hexagonal grids. The solutions are compared with those obtained in analytical case and for rectangular grids. The result for one of the selected hexagonal grids better fits to the analytical solution then the results for the other considered grids. The obtained results may contribute to better understanding of the appropriateness of hexagonal grids in atmospheric and oceanographic modeling and numerical computations.

1. Introduction

According to Arakawa (1970) there are two main problems in numerical simulation of the atmospheric dynamics. One problem is to properly describe the dispersion of high-frequency gravity-inertial waves which is the main mechanism of the geostrophic adjustment process. A study of the influence of different arrangements of the dependent variables in space (both on rectangular and hexagonal grids), on the dispersion properties of the gravity-inertial waves has been considered by various authors (e.g. Arakawa, 1972; Ničković, 1994).

The second problem is the simulation of the quasi-geostrophic flow, established by the geostrophic adjustment process. It is mainly governed by the advection terms of the equations. Restricting, for simplicity the present analysis to two-dimensional shallow-water equations, the advection of vorticity will be considered since it describes the most essential part of these motions.

This article is addressed to the problem of simulation of large-scale atmospheric motions on the hexagonal grids. The analysis of their numerical features, given by numerous authors (Sadourny et al., 1968; Williamson, 1968; Thacker, 1977, 1978; Steppeler et al., 1990; Ničković, 1994), indicate that in many aspects hexagonal grids are at least not inferior to the rectangular ones. Due to their good grid isotropy, hexagonal grids provide correct radial dispersion of gravity waves (Ničković, 1994) and may be a viable alternative to the rectangular grids.

Finally, we compare the analytical form of Rossby wave solution with its numerical analogues evaluated on hexagonal grids, as it was done by Mesinger (1979) and Gavrilov (1985) for various rectangular grids.

2. Continuous Case

2.1 Orthogonal Coordinate System Oxy

Let us consider the system of linearized shallow-water equations in β plane in orthogonal coordinate system Oxy

\[
\frac{\partial u}{\partial t} - \beta y v - f_0 v + g \frac{\partial h}{\partial x} = 0,
\]

(1a)
The symbols used here, as well as those used further in the text, have their usual meaning (e.g. Holton, 1979). Following the procedure from Gavrilov (1985), it is suitable to introduce the quasi-geostrophic equations of motion in order to obtain the frequency of Rossby waves. The time derivatives and terms containing factor \( \beta \) in equations of motion (1a) and (1b) are small in comparison to the other terms. By neglecting those terms, we obtain geostrophic system of equations. The solution of geostrophic system of equations is geostrophic wind

\[
\mathbf{u}_g = -\frac{g}{f_0} \frac{\partial h}{\partial y}, \quad \mathbf{v}_g = \frac{g}{f_0} \frac{\partial h}{\partial x}.
\] (2)

Substituting this solution in previously neglected terms in (1a) and (1b), we obtain the quasi-geostrophic equations of motion

\[
\frac{\partial u_g}{\partial t} - \beta y v_g - f_0 v + \frac{\partial h}{\partial x} = 0,
\] (3a)

\[
\frac{\partial v_g}{\partial t} + \beta u + f_0 u + \frac{\partial h}{\partial y} = 0.
\] (3b)

The geostrophic vorticity is equal to

\[
\zeta_g = k \cdot \nabla \times \mathbf{v}_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{g}{f_0} \nabla^2 h.
\] (4)

The vorticity equation can be evaluated from (3) if we differentiate (3a) with respect to \( y \), (3b) with respect to \( x \) and then subtract the results

\[
\frac{\partial}{\partial t} \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) + f_0 \left( \frac{\partial u_g}{\partial x} + \frac{\partial v}{\partial y} \right) + \beta v_g = 0.
\] (5)

The quasi-geostrophic vorticity Eq. (5) can be further expressed in terms of height using (2) and continuity Eq. (1c), as

\[
\frac{\partial}{\partial t} \left( \frac{f_0^2}{gH} \right) h + \beta \frac{\partial h}{\partial x} = 0.
\] (6)

By substituting the wave-form solution

\[
h = H e^{i(kx + ly - \omega t)}
\] (7)

into (6) we get the frequency of Rossby waves

\[
v = -\frac{\beta k}{k^2 + l^2 + \frac{f_0^2}{gH}}.
\] (8)

It is known that the frequency of Rossby waves in non-divergent one-dimensional case is

\[
v = -\frac{\beta}{k}.
\] (9)

If we compare the frequency (9) with the frequency (8) for one-dimensional case (\( l = 0 \)), we conclude that the term \( \frac{f_0^2}{gH} \) results from the assumption that the divergence is not equal to zero.

2.2 Nonorthogonal Coordinate System \( O\acute{x}y' \)

For further consideration, it is suitable to introduce a nonorthogonal coordinate system \( O\acute{x}y' \) with the angle 120° between axes displayed in Fig. 1. We shall denote by \( (x, y) \) the coordinates of an arbitrary point \( M \) in \( Oxy \) system and by \( (x', y') \) in \( O\acute{x}y' \) system. Analogous notation will be used for all other variables. Coordinates of a point, components of velocity and partial derivatives in two coordinate systems are related by

\[
\begin{align*}
x' &= x + \frac{y}{\sqrt{3}}, & y' &= 2y, \\
u' &= u + \frac{v}{\sqrt{3}}, & v' &= \frac{2v}{\sqrt{3}}.
\end{align*}
\] (10)

Fig. 1. Orthogonal \( Oxy \) and nonorthogonal \( O\acute{x}y' \) coordinate systems