Exact expressions are obtained for the Clebsch-Gordan coefficients of the tensor product of irreducible unitary representations of SL(2, C) of the principal and supplementary series.

The Clebsch-Gordan coefficients of the homogeneous Lorentz group SO(3, 1) [or its covering SL(2, C)] play an important role in the group-theoretical approach in elementary particle physics; in particular, in the theory of relativistic wave equations, in scattering theory, for the parametrization of the S matrix, and in the expression of experimental quantities in terms of amplitudes. The problem of the decomposition into irreducible unitary representations of a tensor product of two irreducible unitary representations of SL(2, C) has been solved by Naimark [1-3]. An exact expression for the Clebsch-Gordan coefficients of SL(2, C) has been found for special cases [4-6]. A general procedure for finding the Clebsch-Gordan coefficients for arbitrary irreducible unitary representations of the principal series has been indicated by Anderson et al. [7]. However, their method is also applicable in the case of decomposition of tensor products of irreducible unitary representations of other series of representations of SL(2, C). In the present communication we use the matrix elements of the irreducible unitary representations of the principal and supplementary series of representations of SL(2, C) found by Ström [11, 12] in conjunction with the method of [7] to calculate explicitly the Clebsch-Gordan coefficients in the decompositions of the tensor products in which irreducible unitary representations of the supplementary series participate.

However, if one takes the Sciarri—Toller matrix elements [13], the expression obtained for the Clebsch-Gordan coefficients contains all the cases (irreducible unitary representation of the principal series multiplied by an irreducible unitary representation of the principal series; irreducible unitary representation of the supplementary series multiplied by an irreducible unitary representation of the supplementary series; and an irreducible unitary representation of the principal series multiplied by an irreducible unitary representation of the supplementary series).

1. The Clebsch-Gordan coefficients for an arbitrary locally compact group are defined as follows [8]. Let $H^{\chi_1} \otimes H^{\chi_2}$ be the space of the tensor product of the irreducible unitary representations $D^{\chi_1}$ and $D^{\chi_2}$ of this group, respectively ($\chi_1$ and $\chi_2$ are parameters that define the representations $D^{\chi_1}$ and $D^{\chi_2}$). Then

$$H^{\chi_1} \otimes H^{\chi_2} = \int H^{\chi} d\mu(\chi),$$

where $\chi$ is the set of parameters that define the irreducible unitary representation $D^\chi$ which acts in the space $H^\chi$, and $\mu(\chi)$ is the measure with respect to which the decomposition is made. To formula (1) there corresponds the decomposition

$$D^{\chi_1} \otimes D^{\chi_2} = \int D^{\chi} d\mu(\chi)$$

of the tensor product $D^{\chi_1} \otimes D^{\chi_2}$ into a direct integral of irreducible representations. We shall denote the elements of an orthonormalized basis in $H^\chi$ by $| \chi, i_\chi \rangle$. The Clebsch-Gordan coefficients $\langle \chi_1, i_{\chi_1}; \chi_2, i_{\chi_2} | \chi_1, i_{\chi_1} ; \chi_2, i_{\chi_2} \rangle \equiv \langle \chi_1, i_{\chi_1} | \chi_2, i_{\chi_2} \rangle$ into a basis of the space $H^\chi$.
2. In particular, the irreducible unitary representations of the principal series of SL(2, C) are defined by the parameters \([\chi]\) \(= (\nu, \rho), \chi_1 = (\nu_1, \rho_1), i_\chi = (JM), i_{\chi_k} = (J_k M_k), i = 1, k = 1, 2\), and Eq. (3) takes the form

\[
|\nu p, J M\rangle = \sum_{J_{\nu p}, J_{M1}, J_{M2}} \langle \nu p, J_{M1}; \nu p, J_{M2} | \nu p, J M\rangle |\nu p, J_{M1}; \nu p, J_{M2}\rangle.
\] (4)

On the other hand, using the operator [10]

\[
P_{J_{M1}, M2}^{\nu p} = \mu(\nu, p) \int d g D_{J_{M1}, M2}^{\nu p} (g) T_g
\]

where \(T_g\) is a quasiregular representation of SL(2, C) and the asterisk stands for the complex conjugate, we can project the vectors of one of the bases in which we are interested onto another:

\[
|\nu p J M\rangle = N_{J_{M1}, M2, J_{M1}, J_{M2}}^{\nu p} \int d g D_{J_{M1}, M2}^{\nu p} (g) T_g |\nu p, J_{M1}; \nu p, J_{M2}\rangle.
\] (6)

Here \(N\) is a normalization coefficient equal to [7]

\[
N_{J_{M1}, M2, J_{M1}, J_{M2}}^{\nu p} = \langle \nu p, J_{M1}; \nu p, J_{M2} | \nu p J M'\rangle^*.
\] (7)

and the operator \(T_g\) acts in the Hilbert space \(H^{\nu_1 \rho_1} \otimes H^{\nu_2 \rho_2}\) of the representation \(D^{\nu_1 \rho_1} \otimes D^{\nu_2 \rho_2}\). Knowing the action of \(T_g\) on the basis vector \(|\nu_1 \rho_1, J_{M1}; \nu_2 \rho_2, J_{M2}\rangle\),

\[
T_g |\nu_1 \rho_1, J_{M1}; \nu_2 \rho_2, J_{M2}\rangle = \sum_{J_{M1}, M\nu p, J_{M1}, J_{M2}} D_{J_{M1}, M\nu p, J_{M1}, J_{M2}}^{\nu p} (g) |\nu_1 \rho_1, J_{M1}, J_{M1}, J_{M2} J_{M2}\rangle.
\] (8)

and using the orthonormalization property of the basis elements, we arrive at the expression

\[
\langle \nu_1 \rho_1, J_{M1}; \nu_2 \rho_2, J_{M2} | \nu p J M\rangle \langle \nu p, J_{M1}, J_{M1}, J_{M2} J_{M2}\rangle^* = \int d g D_{J_{M1}, M\nu p, J_{M1}, J_{M2}}^{\nu p} (g) D_{\nu p, J_{M1}, J_{M1}, J_{M2}}^{\nu p} (g).
\] (9)

We now use the well-known SU(2)-parametrization of SL(2, C), which induces the decomposition [7, 13]

\[
D_{\nu p}^{\nu_1 \rho_1} (g) = \sum_{\nu=-\nu_1}^{\nu_1} R_{\nu \nu_1} (\eta, \psi, 0) d_{\nu p}^{\nu_1 \rho_1} (\xi) R_{\nu_1 \nu} (\psi, \theta, 0),
\] (10)

0 \(\leq \eta \leq 4\pi\), 0 \(\leq \varphi_1 \leq 2\pi\), 0 \(\leq \psi \), \(\theta \leq \pi\), 0 \(\leq \xi < \infty\), where \(R_{MM}\) are the matrix elements of the irreducible unitary representation of the subgroup SU(2) and \(d_{\nu p}^{\nu_1 \rho_1} (\xi)\) are the matrix elements of the unitary representation of a one-parametric subgroup.

With allowance for (10), the desired Clebsch-Gordan coefficients can be expressed in terms of the Clebsch-Gordan coefficients of the subgroup SU(2):

\[
\langle \nu_1 \rho_1, J_{M1}; \nu_2 \rho_2, J_{M2} | \nu p J M\rangle \langle \nu p, J_{M1}, J_{M1}, J_{M2} J_{M2}\rangle^* = \frac{64\pi}{(2J + 1)(2J' + 1)} \frac{J_{M1}, J_{M2} | J M | J_{M1}, J_{M2} | J M'}\langle \nu_1 \rho_1, J_{M1}; \nu_2 \rho_2, J_{M2} | \nu p J M\rangle \langle \nu p, J_{M1}, J_{M1}, J_{M2} J_{M2}\rangle^* \times \sum_{m_1, m_2} \langle J_{M1}, J_{M2} | J_{m1} + m_2, J_{m1}' + m_1 + m_2 \rangle \langle J_{m1}, J_{m1}' | J_{M1} + m_2, J_{M1}' + m_1 \rangle \int d_{\nu p, \nu_1 \rho_1}^{\nu_1 \rho_1} (\xi) d_{\nu p, \nu_1 \rho_1}^{\nu_1 \rho_1} (\xi) d_{\nu p, \nu_1 \rho_1}^{\nu_1 \rho_1} (\xi) \sin \xi d^*_\xi d^*_\xi.
\] (11)


3. An irreducible unitary representation of the supplementary series of SL(2, C) is specified by the parameter \([\chi]\) \(= (\sigma, \sigma), 0 < \sigma < 2\). For the case of a "tensor square" of irreducible unitary representations of the supplementary series of SL(2, C), the decomposition has the form [3]

\[
D^\sigma \otimes D^\sigma = \oplus \sum_{\sigma_1 + \sigma_2 \leq 2} \int d\mu(\nu, \rho) D^\sigma\nu_{\sigma_1 + \sigma_2} \text{ for } \sigma_1 + \sigma_2 \leq 2
\] (12)