PROOF OF THE INVARIANCE OF THE BETHE-ANSATZ
SOLUTIONS UNDER COMPLEX CONJUGATION

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It is shown that the solutions to the systems of algebraic equations obtained when the Bethe ansatz is applied to the integrable XXX and XXZ magnets of arbitrary spin are invariant under complex conjugation.

1. The method of the Bethe ansatz [1-5] makes it possible to reduce the solution of a certain class of problems of one-dimensional mathematical physics to the investigation of a system of algebraic equations. The simplest but fairly characteristic example of such a system of Bethe-ansatz equations is

\[
\left( \frac{\lambda_j - i/2}{\lambda_j + i/2} \right)^N = - \prod_{k=1}^{l} \frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i}, \quad j=1, \ldots, l; \quad l \leq \frac{N}{2},
\]

the solutions of which are to be sought in the form of sets \{\lambda_j\} consisting of pairwise distinct complex numbers \lambda_j (other solutions of the system (1) are not included in the scheme of the Bethe ansatz).

If \{\lambda_j\} is such a solution, then the complex conjugate set \{\overline{\lambda}_j\} is also obviously a solution of the system (1). The aim of the present paper is to prove the less obvious result \{\lambda_j\} = \{\lambda_j\}. The self-conjugacy of the solutions of the Bethe-ansatz equations has not hitherto been proved in the literature. Nevertheless, it has been adopted as a justified hypothesis by virtually all the specialists in this field and served as the basis for all subsequent conclusions concerning the structure of the solutions of the systems of such equations. We note that in the limit \( N \to \infty \) the self-conjugacy hypothesis for \{\lambda_j\} was proved in [6] for a special class of solutions that are of interest from the point of view of physics (describing excitations above an antiferromagnetic vacuum).

In the present paper, we propose a simple proof of the self-conjugacy of all solutions \{\lambda_j\} of the system (1) and a number of other such systems.

2. Our proof proceeds not so much from an analysis of the system (1) as from its "prehistory" in the context of the algebraic Bethe ansatz [2, 3]. We write down relations of the quantum inverse scattering method that lead to the system (1) and are needed for what follows. The original physical model is the one-dimensional isotropic Heisenberg magnet or the XXX model of spin 1/2 with Hamiltonian

\[
H = -\frac{1}{2} \sum_{n=1}^{N} \left( \sum_{i=1}^{3} \sigma_i \sigma_{i+1}^{\dagger} - 1 \right), \quad \sigma_{n+1}^{\dagger} = \sigma_n^{\dagger},
\]

which describes the interaction of \( N \) particles with spin 1/2 on a periodic chain. Here, \( \sigma_n^{\pm} \) are Pauli matrices acting on the space \( V_n \cong \mathbb{C}^4 \) associated with the site with number \( n \). We define the \( 4 \times 4 \) matrix

\[
L_n(\lambda) = \lambda + \frac{i}{2} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i^{\dagger},
\]

which acts on \( V_n \otimes V_n \), where \( V_n \cong \mathbb{C}^4 \) is an auxiliary space, and the transfer matrix \( \tau(\lambda) = tr(L_n(\lambda) \ldots L_1(\lambda)) \), which acts on \( W = \prod_{n=1}^{N} \otimes V_n \). In the same space \( W \), every solution \{\lambda_j\} of the system (1) is associated with a vector \(|\{\lambda_j\}\rangle\), which for any complex \( \lambda \) is an eigenvector for \( \tau(\lambda) \),

\[
\tau(\lambda)|\{\lambda_j\}\rangle = \Lambda(\lambda, \{\lambda_j\})|\{\lambda_j\}\rangle,
\]

with eigenvalue

\[ \Lambda(\lambda, \{\lambda_j\}) = \left(\lambda - \frac{i}{2}\right)^n \prod_{j=1}^{i} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j - i} \prod_{j=1}^{i} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j + i}. \]  

(3)

The vectors \( |\{\lambda_j\}\rangle \) are also eigenvectors for the Hamiltonian \( H \), since

\[ H = -i \frac{d}{d\lambda} \ln \tau(\lambda) \mu/\alpha + N. \]  

(4)

For the precise definition of the vectors \( |\{\lambda_j\}\rangle \) and the derivation of Eqs. (2)-(4), see [2, 3].

We now find the Hermitian-conjugate matrix \( \tau^\dagger(\lambda) \). For this, following [2], we take the Hermitian conjugate of the matrix \( L_n(\lambda) \) only in the space \( V_n \) (i.e., we take the Hermitian conjugates of \( \sigma_i^a \) (which do not change), and of \( \sigma_i^c \) only the complex conjugate, omitting the transposing). Explicitly, \( L_n^+(\lambda) = \lambda - \frac{i}{2} \) \((\sigma_i^a \otimes \sigma_i^a - \sigma_i^c \otimes \sigma_i^c + \sigma_i^c \otimes \sigma_i^c)\), and we readily see that

\[ L_n^+(\lambda) = \sigma_i^a L_n(\lambda) \sigma_i^a. \]  

(5)

From this it immediately follows that \( \tau^\dagger(\lambda) = \tau(\lambda) \), and hence also

\[ \Lambda(\lambda, \{\lambda_j\}) = \Lambda(\lambda^*, \{\lambda_j^*\}). \]  

(6)

This equation is valid for any \( \lambda \) if \( \{\lambda_j\} \) satisfies (1).

3. The relations (3) and (6) are sufficient to prove the self-conjugacy of \( \{\lambda_j\} \). Substituting (3) in (6), then redenoting \( \lambda^* \) by \( \lambda \) and eliminating the denominators, we arrive at

\[
\left(\lambda - \frac{i}{2}\right)^n \left[ \prod_{j=1}^{i} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j - i} \prod_{j=1}^{i} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j + i} \right] f(\lambda) = \left(\lambda + \frac{i}{2}\right)^n f(\lambda - i).
\]  

(7)

Denoting the expression in the square brackets on the left-hand side by \( f(\lambda) \), we have

\[ \left(\lambda - \frac{i}{2}\right)^n f(\lambda) = \left(\lambda + \frac{i}{2}\right)^n f(\lambda - i). \]  

(8)

But by virtue of the condition \( i \leqslant N/2 \), \( f(\lambda) \) is a polynomial in \( \lambda \) whose degree is less than \( N \). Therefore, it cannot contain as a factor \( (\lambda + i/2)^n \), as required by formula (9) and uniqueness of the decompositon of the polynomials into simple factors. The only way out is that for all \( \lambda \) \( f(\lambda) = 0 \), i.e.,

\[ \prod_{j=1}^{i} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j - i} \prod_{j=1}^{i} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j + i} = 0. \]  

(9)

The further arguments repeat those of [6]. If among the set there are some that are real and mutually conjugate, the corresponding factors in (9) cancel. Suppose however that the complete set \( \{\lambda_j\} \) is not exhausted by such \( \lambda_j \), i.e., there exists a set \( \{\lambda_m\} \subset \{\lambda_j\} \) such that \( \{\lambda_m\} \cap \{\lambda_m^*\} = \emptyset \). We label the elements in \( \{\lambda_m\} \) from \( 1 \) to a certain \( i' \leqslant i \). The relation (9) takes the form

\[ \prod_{m=1}^{i'} (\lambda - \lambda_m + i) (\lambda - \lambda_m - i) = \prod_{m=1}^{i'} (\lambda - \lambda_m + i) (\lambda - \lambda_m - i). \]  

(10)

We now choose in \( \{\lambda_m\} \) the element \( \lambda_p \) with the greatest imaginary part. If there are several such elements, we take any one of them. The factor \( (\lambda - \lambda_p) \) on the left-hand side of (10) must correspond to the same factor on the right-hand side. But that factor does not occur among the factors \( (\lambda - \lambda_m) \) by virtue of the definition of \( \{\lambda_m\} \), nor among the \( (\lambda - \lambda_m + i) \) by virtue of \( \text{Im} \lambda_p \geqslant \text{Im} \lambda_m \). We can only conclude that \( i' = 0 \). We have shown that \( \{\lambda_j\} = \{\lambda_i\} \).

A direct consequence of this fact is reality of the eigenvalues of the local integrals of the motion \( I_k \) on the vectors \( |\{\lambda_i\}\rangle \). Indeed [7].