THE FLUCTUATIONS OF THE PARAMETERS OF THE NORMAL WAVE IN SUPERREFRACTION

M. B. Kanevskii

The method of small perturbations was used to obtain expressions for the one-dimensional spectra of the level and phase fluctuations of a normal wave propagating in a randomly inhomogeneous medium in the presence of regular superrefraction. It is shown that, in superrefraction, small amplitude fluctuations increase with distance more slowly than in the case when the average permittivity of the medium does not depend on the coordinates.

The expressions for the fluctuation spectra of the parameters of a plane wave propagating in a randomly inhomogeneous medium with a constant value of $\bar{\varepsilon}$ (the average permittivity) are well known [1].

In the present paper the one-dimensional analogs of the expressions mentioned are derived from the first approximation of the method of small perturbations for the case when the scattering takes place under conditions of regular superrefraction.

1. Let us consider the scalar wave equation

$$\Delta + k^2 \{ \varepsilon(z) + \varepsilon_1(x, y, z) \} \psi = 0,$$

$$\psi_{|z=0} = 0,$$

(1)

where $k = 2\pi / \lambda$ is the wave number; $\varepsilon(z)$ is a monotonically decreasing function of the vertical coordinate. All of the random inhomogeneities of the medium are situated in the domain $x \geq 0$; the field of the quantity $\varepsilon_1$ is statistically uniform; $l \gg \lambda$, where $l$ is the least of the scales of the random inhomogeneities which are essential for the given problem; $(\varepsilon_1^2)^{1/2} = \mu \ll 1$.

From the domain $x < 0$ a normal wave $\psi_0 = f(z) \exp (ik_\parallel x)$ is incident on the boundary $x = 0$. Here $k_\parallel$ is the longitudinal wave number; the altitude multiplier $f(z)$ is a real function which satisfies the one-dimensional wave equation and the condition at $z = 0$:

$$f'' + k^2 s(z) f = 0, \quad s(z) = \varepsilon(z) - k^2 / k^2,$$

$$f(0) = 0.$$

(2)

The function $f(z)$ oscillates in the range $0 < z < z_f$ (the altitude of the reversal point $z_f$ is a root of the equation $s(z) = 0$) and decreases exponentially for $z > z_f$. Let us introduce $\lambda_f = k^{-2/3} |\varepsilon'|^{-1/3}$ which is the scale of the oscillations of $f(z)$ near the reversal point (the derivative $\varepsilon'$ is taken at the point $z_f$). For convenience we assume that the function $f(z)$ is normalized, i.e., $\int_0^\infty f^2(z)dz = 1$.

Let us expand the field $\psi$ into a series $\psi = \psi_0 + \psi_1 + \ldots$, where $|\psi_1| \sim \mu^2 |\psi_0|$, and then, setting the terms of order $\mu$ equal to zero, we obtain the equation for $\psi_1$:

$$[\Delta + k^2 \varepsilon(z)] \psi_1 = -k^2 \varepsilon_1 \psi_0,$$

$$\psi_{|z=0} = 0.$$

(3)
The quantities $\varepsilon_1(x, y, z)$ and $\psi_1(x, y, z)$ are represented in the form of stochastic Fourier–Stiltjes integrals $\varepsilon_1(x, y, z) = \int_{-\infty}^{\infty} \exp(ik_0y) d\varepsilon(x, y, \kappa_2)$, $\psi_1(x, y, z) = \int_{-\infty}^{\infty} \exp(ik_0y) d\psi(x, z, \kappa_2)$, and then we go over from (3) to an equation that couples the spectral amplitudes $d\psi$ and $d\varepsilon$:

$$[\Delta_\varepsilon + k^2\bar{\varepsilon}(z) - \varepsilon_1^2] d\psi = - k^2 \varepsilon_1 d\varepsilon, \\ d\psi|_{z=0} = 0.$$  

Neglecting back scattering, we write the approximate solution of Eq. (4) in the form of an infinite series

$$d\psi(x, z, \kappa_2) = \frac{ik^2}{2} \sum_n \bar{\beta}_n^{-1} u_n(z) \int_0^x dx' \int_0^z dz' [d\psi(x', z', \kappa_2)] f(z') \exp(ik_0 x') u_n(z') \exp[it\beta_n(x-x')]$$  

where $u_n(z)$ is the system of real orthonormalized eigenfunctions of the problem

$$u_n'' + [k^2 \bar{\varepsilon}(z) - \varepsilon_1^2] u_n'' + \beta_n^2 u_n = 0, \\ u_n(0) = 0,$$

that decrease exponentially for large $z$; the quantities $\beta_n^2$ are eigenvalues. It is evident that $u_n(z)$ are none other than the altitude multipliers of normal waves in a regular waveguide corresponding to the case $\varepsilon(z) = \bar{\varepsilon}(z)$. In particular, for $\beta_n^2 = k_1^2 - \chi_2^2$ the solution of the problem is $f(z)$, the altitude multiplier of the primary wave.

By means of the relationships $\beta_n^2 + \chi_2^2 = k_0^2 \bar{\varepsilon}(0) \cos^2 \theta_n$ and $k_1^2 = k_0^2 \bar{\varepsilon}(0) \cos^2 \phi$ we introduce $\theta_n$ and $\phi$, which are the glancing angles of the $n$-th and primary normal waves in the boundary $z = 0$; under these conditions we shall assume that $\theta_n, \phi_n \ll 1$. From the indicated relationships we find $\beta_n^2 = k_0^2 - \chi_2^2 - k_0^2 \bar{\varepsilon}(0) \cdot (\theta_1^2 + \phi_n^2) \delta_f(\theta_1 - \phi_n)$. It should be expected that for scattering of the primary wave by inhomogeneities having a vertical scale $l_0 \sim 1/\chi_2^\max$ basically only those normal waves for which $|\theta_n - \phi_1| \lesssim \chi_2^\max/k$ will be excited; therefore

$$\beta_n^2 = k_1^2 - \chi_2^2 + \varepsilon_n, \quad |\varepsilon_n| \lesssim 2\theta_1^2 k_0^2 \chi_2^\max + (\chi_2^\max)^2 \ll k_1^2.$$  

If the condition

$$|\varepsilon_n - \chi_2^2 x/k_1^2| \ll 1,$$

is fulfilled, it follows that $\beta_n = k_1 + (\varepsilon_n - \chi_2^2)/2k_1$ may be substituted into the argument of the exponential in (5).

Comparing (7) with the relationship $\beta_n^2 + \chi_2^2 = k_0^2 \bar{\varepsilon}(z_n) - \chi_2^2$ derived from (6) ($z_n$ is the height of the rotation point of the normal wave), we find $\sigma_n = k_0^2 \bar{\varepsilon}(z_n) - k_1^2 = k_0^2 \bar{\varepsilon}(z_n) = k_0^2 \varepsilon_n$, therefore

$$\beta_n^2 = k_1 + (k_0^2 \varepsilon_n - \chi_2^2)/2k_1.$$  

If $2\theta_1^2 k_1 \chi_2^\max \gg (\chi_2^\max)^2$, which in the case of isomeric inhomogeneities of the medium corresponds to the condition $\theta_1^2 \gg \lambda/l_0$, it follows from (8) that a constraint is imposed on the distance $x \ll l^2/\lambda \theta_1^2$.

If $2\theta_1^2 k_1 \chi_2^\max \ll (\chi_2^\max)^2$, then we arrive at the well-known (see [1]) constraint $x \ll l^2/\lambda^3$.

2. The solution (5) may be used to find both the mean-square values of the amplitudes of the secondary waves [2, 3] and the fluctuations of the amplitude and phase of the total field.

Let us represent the fluctuation field in the form $\psi_1 = \{(1 + \delta A/f) \exp(iS_1) - 1\} \exp(ik_0 x)$, which is suitable outside the neighborhoods of the zeros of the function $f(z)$, and $\delta A$ and $S_1$ are, respectively, the amplitude and phase fluctuations. Outside the neighborhoods of the zeros of the function $f(z)$ it may be assumed for $S_1 \ll 1$ and $\delta A/f = \chi \ll 1$ that $\psi_1 \approx (\chi + iS_1)f \exp(ik_0 x)$.

Taking account of (9), we write the expression for $d\psi$ (the spectral amplitude of the quantity $\varphi = \chi + iS_1$) which is suitable outside the neighborhoods of the zeros of $f(z)$:

$$d\psi'(x, z, \kappa_2) = \frac{ik^2}{2k_1 f(z)} \sum_n u_n(z) \int_0^x dx' \int_0^z dz' [d\psi'(x', z', \kappa_2)] f(z') u_n(z') \exp \left[ \frac{i}{2k_1} (k_0^2 \varepsilon_n - \chi_2^2) (x - x') \right].$$  

Equation (10) provides the possibility of finding the one-dimensional spectra of the level and phase fluctuations $F_\chi$ and $F_S$. 

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