PLANE INCOMPRESSIBLE FLUID FLOW PAST AN ARRAY OF ARBITRARY PROFILES VIBRATING WITH ARBITRARY PHASE SHIFT

Z. B. Kazimirski, V. V. Nitusov, and G. S. Samoilovich


We consider the problem of the vibration of an array of arbitrary profiles with arbitrary phase shift. Account is taken of the influence of the vortex wakes. The vibration amplitude is assumed to be small. The problem reduces to a system of two integral Fredholm equations of the second kind, which are solved on a digital computer. An example calculation is made for an array of arbitrary form.

A large number of studies have considered unsteady flow past an array of profiles. Most authors either solve the problem for thin and slightly curved profiles or they consider the flow past arrays of thin curvilinear profiles [1].

In [2] a study is made of the flow past an array of profiles of arbitrary form oscillating with arbitrary phase shift in the quasi-stationary formulation. The results are reduced to numerical values. Other approaches to the solution of the problem of unsteady flow past an array of profiles of finite thickness are presented in [3-5] (the absence of numerical calculations in [3,4] makes it impossible to evaluate the effectiveness of these methods, while in [5] the calculation is made for a symmetric profile in the quasi-stationary formulation).

§1. Derivation of basic relations. We consider in the plane of the complex variable $z = x + iy$ an infinite array of arbitrary profiles with period $t$ in an ideal incompressible fluid flow. The flow is steady at infinity to the left of the array. The profiles execute synchronous harmonic oscillations with frequency $\nu$ and with arbitrary constant phase shift $\alpha$ from one profile to another ($\alpha \neq 0$). The oscillation amplitude is assumed to be small.

The solution consists in finding the values of the flow velocity over any contour of the array and in the determination of the forces acting on this profile. The flow velocity $W$ past the cascade of oscillating profiles is made up of the velocity $W_0$ of the basic steady flow (the problem of steady flow is assumed to have been solved) and the perturbed velocity $w$ due to the profile oscillation. The velocities $W_0$ and $w$ are absolute flow velocities, i.e., velocities measured relative to a fixed coordinate system. We denote by $W_0$ and $W$ respectively the velocities of the steady state flow at infinity ahead of and behind the cascade.

We introduce the absolute complex velocity $w = w(z, \tau)$ of the perturbed fluid motion. The perturbed motion complex velocity must have the following properties:

1. The function $w(z, \tau)$ must be periodic in step and time:

$$w(z + imt, \tau) = w(z)e^{-i\omega_m \tau}$$

($m = \pm 1, \pm 2, \ldots$).

Here $\tau$ is time, and $j$ is an imaginary unit which does not interact with the imaginary unit $i$.

2. The function $w(z) \to 0$ as $x \to -\infty$.

In accordance with the Thomson theorem, when the circulation about an oscillating profile changes, a vortex sheet, which is a line of discontinuity of the velocity field, trails from the rear edge (Fig. 1).

Let the contours $L_m, L_{m+1}$ ($m = 1, 2, 3, \ldots$) enclose the corresponding profiles and the wakes behind them (the wakes are assumed to have considerable length, but are finite). We draw the contour $L$, which encloses $n$ profiles with their wakes. After making cuts between the contours $L_{m+1}$ we write the value of the complex velocity $w(z)$ at any point $z \in S$ using the Cauchy formula, and, following [2], as $n \to \infty$ we reduce the integration to integration along only the contour $L$. Then the expression for the complex velocity takes the form

$$w(z) = \frac{1}{\nu} \int_{0}^{\infty} w(\xi) \Phi[z-\xi, z, \alpha, \tau] d\xi = 0. \quad (1.1)$$

Here $w(\xi)$ is the boundary value of the function being represented. The contours $L$ and $L_0$ along which the integration is performed, enclose only the primary profile and the wake behind it. The kernel is defined by the expression [2]

$$\Phi[z-\xi, z, \alpha, \tau] = \frac{\text{ch}[(\pi - \alpha)(\xi - z)/\nu]}{\text{sh}[\pi(\xi - z)/\nu]} - \frac{\text{sh}[(\pi - \alpha)(\xi - z)/\nu]}{\text{sh}[\pi(\xi - z)/\nu]}. \quad (1.2)$$

We break down the contour integral into two integrals along the contours $L$ and $L_0$, respectively. We write the expression for the absolute velocity of the perturbed fluid motion for the points $z \in L$ in the form

$$w(z) = v(z) + v_0(z)$$

(this representation generalizes somewhat the analogous form used in [6]).

Here $v(z)$ is the complex velocity of the perturbed fluid motion at the contour in the relative motion, and $v_0(z)$ is the known local contour motion velocity. We rewrite (1.1) in a different form:

$$v(z) + \frac{1}{\nu} \int_{0}^{\infty} w(\xi) \Phi[z-\xi, z, \alpha, \tau] d\xi = 0.$$  \quad (1.3)$$

We introduce the variables $\sigma$ and $\sigma_0$, which are the coordinates along the arc of the contour $L$ at the points...
\( \zeta \) and \( z \), respectively. The equation for the velocity \( \bar{v}(z) \), the conjugate of \( v(z) \), is obtained from (1.3) by replacing \( v(z) \) by \( \bar{v}(z) \) and \( w(z) \) by \( \bar{w}(z) \). Since

\[
\bar{v}(\zeta) = v(z) e^{-\beta_0 a_0} = v(s_0) \bar{z}'(s_0),
\]

we have

\[
\bar{v}(\zeta) d\zeta = v(s_0) e^{-i\beta_0 a_0} d\sigma_0 = v(s_0) d\zeta.
\]

Here \( \beta_0 \) is the angle between the tangent to the profile at the point and the abscissa axis, and \( v(s_0) \) is the modulus of the perturbed flow velocity. Considering (1.4) and (1.5), we rewrite (1.3) as follows:

\[
v(s_0) + \frac{z'(s_0)}{t_{L}} \int_{L} v(s_0) \Phi[\zeta(z) - z(s_0), a, t] d\sigma + \frac{\bar{z}'(s_0)}{t_{L}} \int_{L} \bar{v}(\zeta) \Phi[\zeta - z, a, t] d\zeta = -\bar{v}(s_0) \bar{z}'(s_0) - \frac{z'(s_0)}{t_{L}} \int_{L} \bar{v}(\zeta) \Phi[\zeta - z, a, t] d\zeta. \tag{1.6}
\]

For the points \( \zeta \in L_{02} \) the vector \( \bar{w}(\zeta) \) may be written in the form

\[
\bar{w}(\zeta) = \bar{w}_n(\zeta) + \bar{w}_s(\zeta)
\]

in the \( n, s \) coordinate system (Fig. 1), where \( \bar{w}_n(\zeta) \) and \( \bar{w}_s(\zeta) \) are the projections of \( \bar{w}(\zeta) \) on the normal and tangent to the wake respectively. Since the vector \( \bar{w}_n(\zeta) \) is perpendicular to the direction of travel around the contour \( L_{02} \),

\[
\int_{L_{02}} \bar{w}(\zeta) \Phi[\zeta - z, a, t] d\zeta = 0. \tag{1.7}
\]

The integral of the normal component around the contour \( L_{02} \) will be zero, since the normal component does not have a discontinuity and is the same on both sides of the cut.

Introducing the independent variables \( \sigma \) and \( s \) for the contour \( L_{02} \) and considering that \( \bar{w}_n(\zeta) \) is perpendicular to the direction of travel around the contour \( L_{02} \), we write the integral (1.7) in the form

\[
\int_{L_{02}} \bar{w}_s(\zeta) \Phi[\zeta - z, a, t] d\zeta = \int_{L_{02}} \bar{w}_s(\zeta) \Phi[\zeta - z, a, t] d\zeta. \tag{1.8}
\]

Here \( \sigma \) and \( s \) are the arc coordinates of the points \( z \) and \( \zeta \) of the contour \( L_{02} \), measured from some fixed point on the contour.

For the calculations on the contour \( L \) it is convenient to introduce the independent variables \( \varepsilon \) and \( \theta \) in place of \( a_0 \) and \( s_0 \), so that \( 0 \leq \theta < 2\pi \) and \( 0 \leq \varepsilon < 2\pi \). The connection between \( s_0 \) and \( \theta \) is established by the relation

\[
d\sigma_0 = \left[ \left( \frac{dz}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2 \right]^\frac{1}{2} d\theta = \Omega(\theta) d\theta. \tag{1.9}
\]

Considering that \( z'(s_0) = x'(\theta) / \Omega(\theta) \), and introducing the new function \( V(\theta) = v(\theta) \Omega(\theta) \) for convenience, we separate in (1.6) the real part with respect to \( t \). Then we obtain

\[
V(\theta) = \int_{0}^{2\pi} V(\varepsilon) \times
\]

\[
\times \text{Re} \left\{ -\bar{v}(s_0) \bar{z}'(s_0) - \int_{L_{02}} \bar{v}(\zeta) \Phi[\zeta - z, a, t] d\zeta \right\}, \tag{1.10}
\]

To simplify the computational scheme without introducing any material error into the final result, we can assume that the wake behind the profile is rectilinear and that it may be replaced by a discontinuity line (Fig. 1). We assume the velocity \( W_0 \) of the steady flow behind the cascade to be constant in both magnitude and direction. Then the magnitude of the velocity discontinuity must be given by

\[
2w_s(\theta) = \frac{1}{W_0} \frac{dU}{d\varepsilon} = \frac{1}{W_0} \frac{dU}{d\varepsilon}, \quad \tau' = \frac{x(\varepsilon) - x(0)}{W_0 \cos \beta_0}. \tag{1.11}
\]

Here \( \Gamma = \Gamma_0 \exp(i \nu t) \) is the unsteady part of the circulation around the primary profile, \( x(0) \) is the abscissa of the exit edge, and \( \nu \) is the oscillation frequency. With account for the above, from (1.10) we obtain in final form the integral equation in terms of the unknown velocity:

\[
V(\theta) = \int_{0}^{2\pi} V(\varepsilon) \Phi(\varepsilon, \theta, a, t) + \delta(0, a, t) \text{d} \varepsilon = F(\theta). \tag{1.11}
\]