METHOD OF CURVED BODIES IN PROBLEMS OF UNSTEADY HYPERSONIC FLOW PAST SLENDER BODIES

V. V. Lunev

Izv. AN SSSR. Mekhanika Zhidkosti i Gaze, Vol. 3, No. 5, pp. 64–72, 1988

The method of curved bodies involves replacing the unsteady flow past a body by steady flow past a different body obtained from the original body by suitable curvature of its form. The idea of the method was proposed by Vetchinkin in 1938 and was first carried out in [13]. Here the authors started from the assumption that the pressure on the body surface is determined only by its local angle of attack.

We know that this method is justified only for circular motion of a slender body with constant velocity within the framework of subsonic or supersonic linearized theory.

It will be shown below that the method of curved bodies is rigorously justified for hypersonic unsteady flow past slender pointed bodies within the framework of the law of plane sections, which is often used to study unsteady flows, for example [3, 4]. Here the idea of the method involves the selection of a body of form such that for uniform translational motion its wake in a stationary, normally intersected plane coincides in time with the wake of the original body.

The general theory is presented for arbitrary bodies, in particular for bodies of the type of slender oscillating wings, but attention is devoted primarily to the motion of a rigid body of rotation. In this case, in the hypersonic approximation (of the type of [4, 5]) the method also extends to slender blunted bodies.

In the general case this method reduces the four-dimensional unsteady problem to a three-dimensional steady problem, which presents no particular difficulty in view of the existence of suitable methods and programs (for example [6]). Here, in contrast with the classical version of the method [1], in the general case the original body is replaced at very moment of time by a one-parameter (with parameter \( t_0 \)) family of curved bodies.

In the case which is most often encountered in practice of slow oscillation of the body surface, when the unsteady component of the solution is small in comparison with the steady component, the small-parameter method is used, which allows us to represent the solution in a simple form with an explicit linear dependence on the parameter \( t_0 \).

The basic notation is: \( L \) is body length, \( \tau_0 \) is the body characteristic relative thickness or angle of attack, \( \alpha_0 \) is the characteristic Strouhal number, \( r_0 \) is the maximal radius of the blunt nose, \( \rho_0 \), \( c_0 \) are the undisturbed medium density and speed of sound, \( V \) and \( M \) are the velocity and Mach number of the center of rotation or of the point \( x_0 \); \( T_0 \) is the characteristic time of the unsteady motion (for example, the period of the oscillation), \( T = L/V \) is the time for the body to pass a fixed plane, and \( \rho_0 V^2 \) is the pressure.

§1. Let the surface of a slender body in the \( x, y, z \) coordinate system attached to the body with origin at the nose, with the \( x \) axis directed basically opposite to the body motion, have the form

\[
f(t, x, y, z) = 0. \tag{1.1}
\]

The center of rotation is located at the point \((x_0, 0, 0)\). We also introduce the \( x', y', z' \) coordinate system with origin at the center of rotation, with the \( x' \) axis directed along the tangent to the center motion and in the opposite direction. We assume that the velocity \( V \) of this point is constant.

The mutual disposition of these coordinate systems is characterized by the angles \( \alpha \) and \( \beta \) between the \( x \) axis and the planes \( y' = 0 \) and \( z' = 0 \) and the angle \( \gamma \) of rotation about the \( x \) axis, and for \( t = 0 \) we set \( \alpha = \alpha_0, \beta = \beta_0, \gamma = \gamma_0 \). The positive directions of the angles are defined by clockwise rotation about the \( z' \), \( y' \), \( x' \) axes.

Finally, we introduce the \( x'', y'', z'' \) inertial coordinate system, whose axis directions coincide with the directions of the corresponding \( x', y', z' \) axes for \( t = 0 \), moving in the direction of negative \( x'' \) with velocity \( V \) whose origin at \( t = 0 \) is at the point coinciding with the projection of the body nose on the \( x' \) axis.

The relative disposition of the last two systems is defined by the analogous angles \( \alpha'', \beta'', \gamma'' \), which are equal to zero for \( t = 0 \).

We further assume that the body introduces small disturbances in the velocity field transverse to the direction of motion, i.e., the following condition is satisfied:

\[
\tau_0 \ll 1, \quad \omega_0 = \Omega L / V \ll 1, \quad \tau_0 = \max(n_\alpha, n_\beta, \alpha, \beta, M^{-1}), \quad \Omega = \max(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\alpha}', \dot{\beta}', \dot{\gamma}').
\]

\[
n_\alpha = f_1 (f_2^2 + f_3^2 + f_4^2)^{-1/2}, \quad n_\beta = -f_2 (f_1^2 + f_3^2 + f_4^2)^{-1/2}.
\]

Here the dots denote derivatives with respect to time. These conditions are satisfied for the motion of a slender body at small angles of attack and with small angular velocities of the order of \( \Omega \sim \tau_0/T_0 \), so that during the characteristic time \( T = L/V \) the increments of all the angles of rotation remain small, of order \( \omega_0 \sim \tau_0 T / T_0 \).

Then, to within quantities of order \((\tau_0 + \omega_0)^2\) we obtain the following formulas for coordinate transformation:

\[
x = x'' + \alpha_0 y'' - \beta_0 z'',
\]

\[
y = y'' + \psi_0 V t a', - \alpha_0 (x'' - z_0) + \gamma_0 z'',
\]

\[
z = z'' - \frac{1}{2} V \psi t a' + \beta_0 (x'' - z_0) - \gamma_0 y'',
\]

\[
x'' = x - \alpha_0 y + \beta_0 z,
\]

\[
y'' = y - \psi_0 V t a' + \alpha_0 (x - x_0) - \gamma_0 z,
\]

\[
z'' = z + \frac{1}{2} V \psi t a' - \beta_0 (x - x_0) + \gamma_0 y
\]

\[
(\alpha_0 = \alpha + \alpha', \beta_0 = \beta + \beta', \psi_0 = \gamma + \gamma') \tag{1.3}
\]

In the inertial system the nose has the coordinates

\[
x_0'' = 0, \quad y_0'' = -\psi_0 x_0 - \frac{1}{2} V t a',
\]

\[
z_0'' = \beta_0 x_0 + \frac{1}{2} V t a'. \tag{1.4}
\]
and in this system the body form (1.1) is
\[ F(t, x'', y'', z'') = f(t, x(x'', y'', z'')), \]
\[ y(x'', y'', z''), z(x'', y'', z'') = 0. \] (1.5)

We note that in (1.8)-(1.5) all the angles depend on time, so that the function \( F \) depends on \( t \) even if the function \( f \) is independent of \( t \). Since \( Vt \sim L \), the terms \( Vt\alpha/2 \) and \( Vt\beta/2 \) in (1.3) have the same order as the other terms. These terms characterize the transverse displacement of the center of rotation resulting from the curvilinear nature of the trajectory.

Under the conditions (1.2) the characteristic transverse velocity of the gas will be of order \( v_0 \sim (T_0 + \omega_0)V \). Now let the body velocity be hypersonic with the Mach number \( M = V/a_\infty \gg 1 \). Then in accordance with the law of plane sections the disturbed longitudinal velocity will be
\[ u_l = V - u \sim (t_0 + \omega_0)^2 V \ll v_0, \] (1.6)
and the flow in the fixed planes \( \xi = x'' - Vt = -Vt_0 \) crossed by the nose at the moments of time \( t = t_0 \) will coincide with the flow for unsteady expansion of a two-dimensional piston following the law
\[ F^*(t, y'', z'') = F(t, V(t - t_0), y'', z'') \quad (t \gg t_0). \] (1.7)

But in accordance with the law of plane sections this piston will be the equivalent for a translational rigid body having, in the inertial coordinate system, the form
\[ F^*(t_0 + x''/V, y'', z'') = F(t_0 + x''/V, x'', y'', z'') = 0. \] (1.8)

Then in the planes \( \xi = -Vt_0 \) the distribution of all the parameters with respect to the variables \( t, y'', z'' \) for the original body (1.1) or (1.3) and for the rigid so-called curved body (1.8) will be the same.

For a fixed moment of time \( t = T = L/V \) the distribution of the flow parameters along the length of the original body may be obtained by considering the curved bodies corresponding to the various values of \( t_0 = (L - x'')/V \).

If the solution obtained as a result of considering a group of curved bodies has the form \( p^*(t_0, x'', y'', z'') \), then the desired solution for the unsteady motion of the original body for \( t = T \) will have the form
\[ p(T, x, y, z) = p^*(L - x''/V, x'', y'', z''). \] (1.9)

The variables \( x'', y'', z'' \) on the right may be replaced by \( x, y, z \) using formulas (1.3). Here \( p \) is pressure, but the solution as a whole has the same form, of course.

Note that the time interval \( \Delta t = T \) is the minimal time during which the body must be observed to obtain its characteristics at the end of this time segment.

Thus the curved-body method reduces the calculation of the unsteady flow past a body with four independent variables to a group of three-dimensional steady state calculations, whose number or the number of values of \( t_0 \) will be determined by the desired detail in the distribution of the parameters along the length of the original body. Although information at only a single section is used from each such calculation to find the solution at the moment \( t = T \), the data at the other sections are not superfluous and are needed for finding the solution at other moments of time.

In the following we shall consider only the special case of unsteady motion of rigid bodies of revolution:
\[ y^2 + z^2 = r^2(x) \quad (y \sim z \sim r \ll r_0), \] (1.10)

Then we can set \( \gamma_0 = 0 \), and (1.3) takes the form
\[ x'' = x = x' + x_0, \]
\[ y = y'' + \gamma \frac{V}{2} Vt + (x - x_0) \alpha z, \]
\[ y'' = y - \gamma \frac{V}{2} Vt + (x - x_0) \alpha z, \]
\[ z = z'' - \gamma \frac{V}{2} Vt + (x - x_0) \beta z, \]
\[ z'' = z + \gamma \frac{V}{2} V t - (x - x_0) \beta z \] (1.11)

This implies that the curved equivalent body is obtained from the original body by curving and rotating its \( y = z = 0 \) axes in accordance with the law
\[ y'' = -\frac{1}{2} V t [\alpha - \alpha(t_0)] + \frac{1}{2} x [\alpha^{'} + \alpha(t_0)] - x_0 [\alpha - \alpha(t_0)], \]
\[ z'' = \frac{1}{2} V t [\beta - \beta(t_0)] - \frac{1}{2} x [\beta + \beta(t_0)] + x_0 [\beta - \beta(t_0)], \]
\[ (\alpha = \alpha(t_0) + x/V) \ldots \] (1.12)

The transverse sections retain their circular form, and the curved bodies which are equivalent to the original body take the form
\[ (y'' - y_0'')^2 + (z'' - z_0'')^2 = r^2. \] (1.13)

In (1.12) the forward point of the axis of the curved body is shifted for convenience to the \( x'' \) axis, which obviously has no effect on the flow characteristics past the body.

If \( p^*(t_0, x'', r, \varphi) \) is the solution for the curved bodies in the local polar coordinate system
\[ y = y'' - y_0'' = r \cos \varphi, \]
\[ z = z'' - z_0'' = r \sin \varphi, \] (1.14)
then in accordance with (1.9) the solution for the unsteady motion of the original body will have the form
\[ p(T, x, r, \varphi) = p^*(\frac{L - x''}{V}, x, r, \varphi). \] (1.15)

The calculation of the over-all transverse forces and moments reduces to simple integration (with suitable weight) of the pressure distribution (1.15) over the original body, and we shall not write out these equations here.

In conclusion we note that in the theory presented above the body surface may have either a positive or negative break in the generating line with the formation of internal compression shocks, with the limitation, however, that the flow be everywhere separation-free and hypersonic.