DIFFUSION OF SUSPENDED PARTICLES IN AN ISOTROPIC TURBULENCE FIELD

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A model for turbulent motion is proposed which makes it possible to evaluate the pulsation characteristics and the diffusion coefficients of the dispersed phase and also makes it possible to describe the effect of the suspended particles on the turbulence of the suspending medium. Specific calculations are made for the situation when the undisturbed turbulent field is isotropic.

The diffusion of an admixture having inertia in a turbulent stream has been studied previously on the assumption that the three-dimensional turbulence characteristics have practically no effect on the behavior of the suspended particles, so that the random motion of the latter is described by ordinary differential equations containing the natural independent variable—the motion travel time [1–4]. In many cases this assumption is incorrect and the corresponding theory is obviously deficient. For example, a fundamental result of this theory, asserting that the turbulent diffusion coefficients of the particles and of the fluid moles are equal for a long diffusion time, is obviously incorrect if the relative motion of the particles is significant [5].

§1. Turbulent motion model. Let us define the velocity of a turbulent homogeneous fluid by averaging over "physical" volumes of linear dimension \( L \). We then have \( \mathbf{v}_L = \langle \mathbf{v} \rangle + \mathbf{v}_L' \), where \( \langle \cdot \rangle \) denotes ensemble averaging. If \( L \ll \lambda \), where \( \lambda \) is the spatial turbulence micro scale, then \( \mathbf{v}_L \) practically coincides with the true random fluid velocity, while \( \mathbf{v}_L' \) is the true pulsation velocity. However, if \( L \gg \Lambda \), where \( \Lambda \) is the scale of the energy-containing vortices (spatial turbulence macro scale), then \( \langle \mathbf{v}_L' \mathbf{v}_L' \rangle \sim L^3 \) and \( \mathbf{v}_L \) practically coincides with the average velocity \( \langle \mathbf{v} \rangle \). The latter corresponds to accounting only for the largest-scale components of the turbulent vortices; the small-scale details of the motion in the case of this averaging are essentially smoothed out (see discussion in [6]), where the analogous averaging is used in the analysis of specific pseudoturbulent motions in dense dispersive systems.

The small-scale pulsation, which disappear during averaging, make a definite contribution to the total momentum transport in the system and in this regard are similar to the molecular motions which give rise to the phenomenon of the conventional viscous stresses. Let us represent the turbulent stresses due to these pulsations in the form

\[
\mathbf{v}_i^{(L)} := \langle \mathbf{v}_i \rangle \mathbf{L}_i + \mathbf{v}_i^{(L)} + \mathbf{v}_i' \mathbf{L}_i + \mathbf{v}_i' \mathbf{L}_i, \]

\[
\mathbf{v}_i^{(L)} = \mathbf{d}_i \mathbf{v}^{(L)}, \quad \mathbf{d}_i := 3 a_i. \tag{1.1}
\]

Here \( a_i \) is the fluid density and \( \mathbf{v}_i^{(L)} \) is the kinematic turbulent viscosity tensor (diffusion tensor), which depends on the scale \( L \). In particular, as \( L \to \infty \) it becomes the conventional diffusion tensor \( \mathbf{v}_i^{(L)} \) while for \( L \ll \lambda \) we have \( \mathbf{v}_i^{(L)} \approx 0 \).

The Reynolds equations, obtained in the usual way after averaging the Navier-Stokes equations, have the form

\[
\left( \frac{\partial}{\partial t} + \langle \mathbf{v} \rangle \cdot \frac{\partial}{\partial x} \right) \langle \mathbf{v} \rangle = - \frac{\partial}{\partial t} \langle \mathbf{p} \rangle + \frac{1}{d} \cdot \frac{\partial}{\partial x} \langle \mathbf{v}_L \rangle + \langle \mathbf{v}_i^{(L)} \rangle + g = \langle \mathbf{P}_f^{(L)} \rangle, \quad \mathbf{P}_f^{(L)} := \mathbf{P}_f(L) := \mathbf{v}_L \cdot \mathbf{v}_L' \mathbf{L}_i \mathbf{L}_i. \tag{1.2}
\]

Here \( g \) is the acceleration of the external mass field, \( \mathbf{d}_i \) is the pressure, \( \tau_3 \) is the conventional viscous stress tensor in a fluid with the viscosity \( \mu_0 \), and \( -\mathbf{d}_i \mathbf{P}_f \) is the Reynolds stress tensor due to the sufficiently large-scale pulsations. Obviously, \( \mathbf{P}_f^{(L)} \to 0 \) as \( L \to \infty \) and \( -\mathbf{d}_i \mathbf{P}_f^{(L)} \) becomes the conventional Reynolds stress tensor as \( L \to 0 \).

We note that even beginning with the familiar Boussinesq hypothesis, relations of the type (1.1) in [L \to \infty] have been under repeatedly in semi-empirical turbulence theories. The various representations for \( \mathbf{v}_L \) which have been postulated previously are discussed critically in [2]; they all have many defects of a fundamental nature. In particular, the need to satisfy the moment of momentum conservation law (symmetry of the tensor \( \tau \)) imposes certain constraints on the values of the turbulent diffusion tensor components, and the significance of these constraints remains vague. Additional inconsistencies arise in convoluting the tensor \( \tau \) and in using the continuity equation [2]. To some degree all these deficiencies are also characteristic of the model in [6]. It is easy to see that the proposed model is free of these defects: it does not require the introduction of a viscosity tensor of fourth rank, is easily obtained after arguments which are usual in semi-empirical theories, and conserves moment of momentum is allowed for.

We note also that the breakdown assumed in (1.1) and (1.2) of the total turbulent stresses into Reynolds and viscous parts is an extension of the familiar Heisenberg hypothesis [7, 8] on the spectral transport of turbulent energy. However, as we shall see in the following, our model leads to a quite definite functional dependence of \( \mathbf{v}_f \) on the spectral functions, i.e., it makes it possible to choose between the various mathematical formulations of the Heisenberg hypothesis which have been previously proposed (a thorough survey of the various forms of the dependence of \( \mathbf{v}_f \) on the turbulence spectral characteristics is given in [8]).

Subtracting (1.2) from the Navier-Stokes equations and considering (1.1), we obtain the equation for the pulsations \( \mathbf{v}_L' \):

\[
\left( \frac{\partial}{\partial t} + \langle \mathbf{v} \rangle \cdot \frac{\partial}{\partial x} \right) \mathbf{v}_L' + \langle \mathbf{v}_L' \mathbf{v}_L' \rangle = - \frac{\partial}{\partial t} \mathbf{d}_i \mathbf{P}_f + \left( \mathbf{v}_L' \cdot \frac{\partial}{\partial x} \right) \mathbf{v}_L' + \frac{\partial}{\partial x} \langle \mathbf{P}_f^{(L)} \rangle - \mathbf{P}_f^{(L)}, \quad \mathbf{v}_0 = \frac{\mu_0}{\mathbf{d}_i}. \tag{1.3}
\]
Given the initial random field \( v_{i}^{L}(t_{0}, \mathbf{r}) \), Eqs. (1.3) describe, first, the degeneration of this fluctuation field with time and, second, the development of a new fluctuation caused by the action of the random force in the last term of the right side of (1.3). We set
\[
\frac{\partial}{\partial \tau} \langle F_{i}^{+} \rangle = - \frac{1}{2} \rho \frac{\partial}{\partial \tau} \langle F_{i}^{+} \rangle + \frac{1}{2} \frac{\partial}{\partial \tau} \rho - F_{i}^{+},
\]
(1.4)
where \( F_{i}^{+} \) is a random function of the coordinates and time. The characteristic time \( F_{i}^{+} \) is of the order of the turbulence "micro time scale" \( \tau \), analogous in meaning to the scale \( \lambda \) and representing the characteristic lifetime of the smallest eddies. Consequently, the characteristic time for degeneration of the initial fluctuation \( v_{i}^{L}(t_{0}, \mathbf{r}) \) defines the turbulence "macro time scale" \( T \), having the same order of magnitude as the lifetime of the energy-containing eddies. It is clear that \( \tau \ll T, \lambda \ll \lambda ; \) consequently, neglecting the processes taking place during the time \( \tau \), the quantity \( F_{i}^{+} \) may be considered a Markov random time function.

The idea of introducing some additional random terms into the regularized equations of motion is not new. An idea of this type has been used previously in the theory of electromagnetic field fluctuations in a continuum [5]; the possibility of introducing "foreign" locally independent random forces and thermal fluxes into the equations of hydrodynamics is noted by Landau and Lifshitz in [10]. This idea has been applied to turbulent motion theory by Novikov [11, 12] and Edwards [13], for example. The proposed model extends the heuristic approach of [11–13], in which very simple stochastic relations of the Langevin and Onsager equation type were used, and at the same time it gives a clear-cut physical meaning to the quantities appearing in these relations. Physically, the introduction of the random quantity (1.4) into Eq. (1.3) is in a certain sense equivalent to the introduction of noise in [6].

Note that (1.3) together with the continuity equation makes it possible in the general case to express the complete tensor \( v_{i}^{L} \) of the spectral density of the process in the form of a function of the pulsation frequency \( \omega \) and of the partial spectral density tensor (energy spectrum tensor), defined by simultaneous correlation functions. The latter yields the possibility of expressing \( v_{i}^{L}(\omega) \) or \( v(k) \), where \( k \) is the wave number corresponding to the scale \( L \), in the form of a functional of the energy spectrum tensor and thereby makes possible the closure of the turbulent motion equations. In fact, using this functional in the equation for the turbulence energy spectrum we obtain a nonlinear integro-differential tensor equation defining this spectrum. The application of the model to the analysis of turbulence in various flows and the calculation of the corresponding energy spectra is of independent interest. Here we simply note that preliminary results concerning the form of the isotropic turbulence spectrum and the spectrum of turbulence with transverse shear are very promising.

In conclusion we note that the essence of the proposed method consists in specifying the type of influence of random waves with different wave numbers \( k \) on the behavior of a wave with a definite wave number \( k \). The action of waves with \( k \approx k \) leads to quasistatic dissipation of the energy of the wave in question by means of transfer of energy to the small-scale disturbances, while the action of waves with \( k \approx k \) shows up in the occurrence of the familiar Reynolds turbulence stresses. A further development of this model is also possible, in which an analogous distinction would be made between the effect of waves with different frequency \( \omega \) on the wave of frequency \( \omega \). This extension may be made by introducing into the equations the turbulent viscosity tensor corresponding to a finite diffusion time which depends on the frequency \( \omega \), rather than that corresponding to an infinite diffusion time. The advantages of such a generalization can be judged, naturally, only after comparing the results based on the proposed model with experimental data.

## 2. Turbulent motions of a dilute dispersed system

In the following we consider a system of non-interacting particles which are suspended in a turbulent fluid, supposing that the volumetric concentration of particles in the mixture \( \rho \ll 1 \) and that their radius \( a < \lambda \). We take the following expression for the interaction force of single particles with the liquid phase per unit mixture volume:
\[
d_{dL} = - \rho \left( \frac{\partial}{\partial \tau} + \frac{1}{2} \frac{\partial}{\partial \tau} \rho \right) + \frac{1}{2} \frac{\partial}{\partial \tau} \rho - F_{i}^{+}
\]
(2.1)
Here \( s \) and \( s' \) are constants of order unity (for Stokes-type motion of rigid spheres \( s = s' = 1 \)), \( w_{L} \) is the velocity of the dispersed phase, and differentiation with respect to time in (2.1) is carried out along the streamlines of this phase. The terms in the right side of (2.1) describe the forces associated with the pressure gradient (usually also turbulent) and the acceleration of the added fluid mass, the linear viscid resistance to relative particle motion, and the unsteady inertial force describing the influence of the motion prehistory (Basset force).

The equations of the pulsational phase motion are obtained similarly to (1.3). Considering in essence only single particles, we neglect in the following the concentration pulsations \( \rho_{L}^{'} \) (\( \rho_{L} = \langle \rho \rangle = \rho \)). Considering for simplicity the characteristic scales of the mean flow parameters large in comparison with the turbulence scales (so that we can neglect the second term in the left side of (1.3) and the analogous terms in (2.1)), we obtain the momentum and mass conservation equations in the pulsation motion in the form
\[
\begin{align*}
\frac{D}{Dt} v_{i}' - \frac{\rho}{\rho} \frac{\partial}{\partial \tau} v_{i}' &= - \frac{\rho}{\rho} \frac{\partial}{\partial \tau} v_{i}' + F_{i} + \\
&+ \left( \left( \frac{1}{2} \frac{\partial}{\partial \tau} + \frac{1}{2} \frac{\partial}{\partial \tau} \rho \right) \left( \frac{\partial}{\partial \tau} + \frac{1}{2} \frac{\partial}{\partial \tau} \rho \right) \right) v_{i}' - \\
&- \frac{1}{2} \rho \frac{D}{Dt} (v_{i}' - w_{L}') - \frac{\rho}{\rho} c (v_{i}' - w_{L}') - \\
&- \frac{\rho}{\rho} c' \frac{D}{Dt} (v_{i}' - w_{L}') - t' + \\
&+ \left( \left( \frac{1}{2} \frac{\partial}{\partial \tau} + \frac{1}{2} \frac{\partial}{\partial \tau} \rho \right) \left( \frac{\partial}{\partial \tau} + \frac{1}{2} \frac{\partial}{\partial \tau} \rho \right) \right) w_{L}' + \frac{\rho}{\rho} (v_{i}' - w_{L}') + \frac{\rho}{\rho} (v_{i}' - w_{L}') \cdot \frac{\partial}{\partial \tau} - \\
&- \frac{1}{2} \rho \frac{D}{Dt} (v_{i}' - w_{L}') - t' + \\
&= u = \langle v \rangle - \langle w \rangle,
\end{align*}
\]
(2.2)