Synchronization of a self-excited oscillator by an external harmonic voltage is discussed. The method of slowly varying amplitudes is used, in the case of small frequency deviation. The synchronization conditions are shown to be determined by the distribution of the real positive roots of four algebraic equations. The form of these equations depends on a nonlinear function. As an example, we examine the synchronization of an oscillator whose nonlinear element is approximated by a seventh degree polynomial.

Synchronization of a self-excited oscillator by an external harmonic voltage has often been discussed in the literature [1-3, 5]. While the main features of the synchronization phenomenon have been revealed, specially in the cases where the oscillator operates under soft and hard excitation conditions, each of the publications quoted deals with a separate particular case of synchronization, so that a complete picture of the phenomenon can only be obtained from an acquaintance with all these publications.

In addition, nothing has been published on synchronization of an oscillator operating under hard excitation of a complex type. It seems worth discussing all the possible synchronization modes from a unified viewpoint in a single paper.

By analogy with [3], we shall assume that the oscillator to which the external harmonic voltage is applied is described by the equation

\[ \frac{d^2 x}{dt^2} + x = \frac{d}{dt} f(x) + 2\xi \cos \tau. \]

where \( \xi = (\omega^2 - \omega_0^2)/\omega^2 \) is twice the deviation between the frequency \( \omega \) of the external voltage and the oscillator natural frequency \( \omega_0 \). The dimensionless amplitude of the external voltage, and \( f(x) \) is a nonlinear function.

The deviating \( \xi \) and function \( f(x) \) will be assumed to be small.

Elementary working shows that, if we seek the solution of (1) in the form

\[ x = z_1 e^{-i\tau} + z_2 e^{i\tau}, \]

\[ x = j(z_1 e^{-i\tau} - z_2 e^{i\tau}), \]

where \( z_1 \) and \( z_2 \) are slowly varying complex conjugate amplitudes, we can write the abbreviated equation (omitting the complex-conjugate equation)

\[ 2j\dot{z}_1 = jx(z_1, z_2) - i z_1 - z, \]

where

\[ x(z_1, z_2) = \frac{1}{2\pi j} \int_{|z| = 1} \left( \frac{z_1}{w} + \frac{z_2}{\bar{w}} \right) dw = \sum_i \text{res} \left( \frac{z_1}{w_i} + \frac{z_2}{\bar{w}_i} \right), \]

are the singular points of the function \( f(z_1/w + z_2w) \), located in the unit circle, and the summation is over all the residues.

If, in some range of its argument \( x \), we can write \( f(x) \) as a convergent series: \( f(x) = \sum a_n x^n \),

the integrand in (3) will be expressible as the Laurent series

\[
\tilde{f} \left( \frac{z_1}{w} + \frac{z_2}{w} \right) = \sum_{n} a_n \left( \frac{z_1}{w} + \frac{z_2}{w} \right)^n = \sum_{n} a_n \frac{z_1^n z_2^n}{w^{2n}}.
\]  \( \text{(4)} \)

Only those terms of the series (4) for which the condition \( 2m - n = 1 \) is satisfied, i.e., the terms with odd \( n \), will yield a nonzero residue. Putting \( n = 2\nu + 1 \) \((\nu = 0, 1, 2, \ldots)\), we obtain \( m = \nu + 1 \), and hence

\[
x(z_1, z_2) = \sum_{\nu} \frac{a_{2\nu+1}}{(2\nu+1)!} \cdot \frac{1}{\nu!} (z_1 z_2).
\]  \( \text{(5)} \)

If we transform to real variables \( \rho \) and \( \varphi \) by means of the substitutions \( z_1 = \rho e^{j\varphi} \) and \( z_2 = \rho e^{-j\varphi} \), where \( \rho \) and \( \varphi \) may be interpreted physically as the amplitude and phase respectively of the forced oscillations, we obtain (2) in the new form

\[
\begin{align*}
\dot{\rho} & = \varepsilon F(\rho^2) + \varepsilon \sin \varphi, \\
\dot{\varphi} & = \varepsilon \rho + \varepsilon \cos \varphi.
\end{align*}
\]  \( \text{(6)} \)

The equilibrium states of the system (6) are given by the equations

\[
\begin{align*}
\rho_0 F(\rho_0^2) + \varepsilon \sin \varphi_0 & = 0, \\
\varepsilon \varphi_0 + \varepsilon \cos \varphi_0 & = 0.
\end{align*}
\]  \( \text{(7)} \)

If we eliminate the phase \( \varphi_0 \) from this system, and write for simplicity \( \rho^2 = K \), we find that the equation of the amplitude-frequency characteristics is

\[
\varepsilon^2 + F^2(K) = \varepsilon^2/K.
\]  \( \text{(8)} \)

The type and stability of the equilibrium state are determined by the roots of the characteristic equation of the system (6), linearized in the neighborhood of the equilibrium state.

This characteristic equation is

\[
p^3 + 2(F + KF')p + F(F + 2KF') + \varepsilon^2 = 0.
\]  \( \text{(9)} \)

We see from (9) that:

- the equilibrium states are of the saddle type when
  \[
  F(F + 2KF') + \varepsilon^2 < 0;
  \]  \( \text{(10)} \)

- if the reverse inequality to (10) is satisfied, and
  \[
  \varepsilon^2 > K^2(F')^2,
  \]  \( \text{(11)} \)

the equilibrium states are of the focus type;

- when the reverse inequalities to (10) and (11) are satisfied, the equilibrium states are of the node type;

- the equilibrium states of the focus and node type are stable when