AXISYMMETRIC OSCILLATIONS OF TWO SPHERICAL SHELLS IN A COMPRES-SIBLE FLUID

A. N. Guz and L. S. Pal’ko


To find the interaction between spherical shells at the frequency of their free oscillations in a fluid, we examine the problem of axisymmetric oscillations of two identical spherical shells under the assumption that the shell centers of curvature do not coincide. The solution is found for the cases of a compressible and an incompressible fluid by the series method with reduction to an infinite system of linear equations. A mathematical justification of the method used is presented.

§1. Problem formulation. Let the Oxyz Cartesian coordinate system and the Ojxjyjzj local coordinate system be fixed with the spheres as shown in Fig. 1.

(Each sphere is assigned its own subscript, j, j = 1, 2.) In addition, each sphere is assigned the rj, 0j, ~j (j = 1, 2), local spherical coordinates, in which the surface of the j-th sphere is given by the equation rj = R. The distance between the centers of the two spheres is denoted by l. The Oz-axis of the Oxyz system is the axis of symmetry of the shells. It is obvious that with these introduced coordinate systems the problem of axisymmetric oscillations of spherical shells in a compressible fluid reduces to the solution of the following system of equations:

**a)** the equations of elastic oscillations of thin non-shallow spherical shells [1],

\[
(\nabla^2 + 1)^2 + \frac{12(1 - \nu^2)}{k^2h^2} \left( 1 + \frac{\rho_0}{k^2E} \frac{\partial^2}{\partial t^2} \right) \times \nabla^2 + 2 \psi_j = \frac{12\rho_0(1 - \nu^2)(1 + \nu)}{k^2h^2E} \times \nabla^2 + 2 \psi_j - \frac{12\rho_0(1 - \nu^2)}{k^2}\frac{\partial^2 \psi_j}{\partial t^2} = 0,
\]

\(k = \frac{1}{R}, \ j = 1, 2\);

**b)** the equation for the perturbed velocity potential of the fluid,

\[
\frac{4}{r_j^2 \sin \theta_j} \left( \frac{\partial}{\partial r_j} \left( r_j^2 \sin \theta_j \frac{\partial}{\partial r_j} \right) + \frac{\partial}{\partial \theta_j} \left( \sin \theta_j \frac{\partial}{\partial \theta_j} \right) \right) \Phi - \frac{4}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0,
\]

\[
\nabla^2 = \frac{4}{\sin \theta_j} \left( \frac{\partial}{\partial \theta_j} \left( \sin \theta_j \frac{\partial}{\partial \theta_j} \right) \right),
\]

\[
\psi_j = \frac{k}{Eh} (\nabla^2 + 1 - \nu) (\nabla^2 + 2) \psi_j.
\] (1.1)

Here c is the speed of sound in the undisturbed fluid, \(w_j\) is the deflection, t is time, E and \(\nu\) are the modulus of elasticity and the Poisson ratio, respectively, h is the shell thickness, \(\rho_0\) is the density of the shell material, and \(\rho\) is the fluid density.

The solution of (1.2) must satisfy the following boundary conditions at the surface of each sphere [2]:

\[
\Phi(r, \theta, t) = \psi(r, \theta) \sin \omega t = \sum_{j=1}^{\infty} \psi_j(r_j, \theta_j) \sin \omega t,
\] (2.1)

\[
\psi_j(r_j, \theta_j) = \sum_{n=0}^{\infty} a_n^{(j)} h_n^{(j)} (\lambda r_j) P_n(\cos \theta_j)
\]

\[
\lambda = \frac{\omega}{c},
\]

(2.2)

with the coefficients \(a_n^{(j)}\), to be determined later from the boundary conditions (1.3); here \(P_n\) are Legendre polynomials, and \(h_n^{(j)} (\lambda r)\) is defined by the relation

\[
h_n^{(j)} (\lambda r) = \frac{2}{\pi^{1/2}} \frac{H_n^{(j)} (\lambda r)}{\lambda_n^{1/2}}.
\] (2.3)

To find \(a_n^{(j)}\) we represent the functions \(\psi_j\) in the form of expansions analogous to (2.2) in terms of the eigenfunctions of the Legendre equation

\[
\psi_j(\theta_j, t) = \psi_j(\theta_j) \cos \omega t = \cos \omega t \sum_{n=0}^{\infty} b_n^{(j)} P_n(\cos \theta_j)
\]

(2.4)

and we consider oscillations which are symmetric about the plane \(z = 0\).
Then, as a result of the properties of the Legendre polynomial, we find that
$$a_n^{(2)} = (-1)^n a_n^{(1)}, \quad b_n^{(2)} = (-1)^n b_n^{(1)}$$
$$a_n^{(0)} = a_n, \quad b_n^{(0)} = b_n.$$

The expansions (2.1) and (2.4) now take the form

$$q(r, \theta) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \left[ \alpha_n^{(1)} h_n^{(1)}(\lambda r_1) P_n(\cos \theta_1) + (-1)^n h_n^{(1)}(\lambda r_2) P_n(\cos \theta_2) \right],$$
$$\psi_1(\theta_1) = \sum_{n=0}^{\infty} \beta_n P_n(\cos \theta_1), \quad (2.5)$$
$$\psi_2(\theta_2) = \sum_{n=0}^{\infty} (-1)^n \beta_n P_n(\cos \theta_2). \quad (2.6)$$

The wave functions appearing in (2.5) are written in the coordinates of the first sphere; to write them in the coordinates of the second sphere, we use the addition theorem for spherical wave functions [3],

$$h_n^{(1)}(\lambda r_1) P_n(\cos \theta_1) = \sum_{q=0}^{\infty} Q_{q\theta_1} Q_{q\theta_2}(r_1, \theta_1)(r_2, \theta_2) P_q(\cos \theta_2). \quad (2.7)$$

On the basis of (2.7) the potential \( \Phi(r, \theta, t) \) is written as

$$\Phi = \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} \left[ (-1)^q \alpha_n^{(1)} h_n^{(1)}(\lambda r_2) + \sum_{n=0}^{\infty} \alpha_n Q_{q\theta_1}(l, \pi) j_q(\lambda r_2) P_q(\cos \theta_2) \sin \omega t \right]. \quad (2.8)$$

If we now satisfy the boundary conditions (1.3) at the surface of the second sphere, then on the basis of (2.8), (2.4), and (2.6), and also the orthogonality property of the Legendre polynomials, we obtain an infinite system of linear equations for determining the coefficients \( a_n \),

$$x_q + \sum_{n=0}^{\infty} A_{qn} x_n = \beta_q b_q = B_q,$$  

$$a_n = x_n, \quad q = 0, 1, 2, \ldots,$$

$$A_{qn} = (-1)^q \frac{\alpha_n}{h_0^{(1)}} Q_{q\theta_1}(l, \pi),$$

$$\alpha_n = \frac{d}{dr_2^2} j_n(\lambda r_2) |_{r_2 = 1} = j_n'(\lambda r),$$

$$\beta_q = \frac{k \omega [q(q + 1) + v - 1]}{E \alpha_0 h_0^{(1)}(\lambda r)}.$$  

which is quasi-regular, and this means it is uniquely solvable by the truncation method [4]. This statement may be proved as follows.

Using the estimate in [5], we obtain

$$\sum_{n=0}^{\infty} |A_{qn}| < C_1 \sum_{n=0}^{\infty} \frac{(n + q)!}{n!q!} \left( \frac{R}{l} \right)^{q+n}$$

$$\left( C_1 = \text{const} \right). \quad (2.10)$$

Consequently,

$$\sum_{n=0}^{\infty} |A_{qn}| < C_1 \sum_{n=0}^{\infty} \frac{(n + q)!}{n!q!} \left( \frac{R}{l} \right)^{q+n} = C_1 \frac{l}{1 - R} \left( \frac{R}{l} \right)^q. \quad (2.11)$$

Since \( l > 2R \), this series converges for any fixed \( q \), and, therefore, there must be a row number \( q_0 \) beginning with which

$$\sum_{n=0}^{\infty} |A_{qn}| < 1. \quad (2.12)$$

The matrix composed from the right-hand sides of (2.10) in terms of \( q \) and \( n \) obviously forms a completely continuous form in the Hilbert space \( l_2 \), since for it the condition

$$\sum_{n=0}^{\infty} \left[ \frac{(n + q)!}{n!q!} \left( \frac{R}{l} \right)^{q+n} \right]^2 < \infty \quad (2.13)$$

is satisfied if \( l > 2R \), i.e., the spheres do not touch.

It follows from physical considerations that the shearing force is continuous in \( \theta \). Therefore \( w \) has a continuous third derivative, and the function \( \psi \) has a continuous seventh derivative.

Therefore, \( b_q \) is of order \( b_q \sim 1/q^5 \). Now, using the formulas asymptotic in \( q \) for the functions \( j_q(l) \), \( h_q^{(1)}(l) \) we can show that

$$B_q < \text{const} \cdot q^{-5}. \quad (2.14)$$

The right-hand sides of (2.14) are elements of the space \( l_2 \), since the infinite series converges with the common term \( |q^{-5}|^2 \). Thus, system (2.9), with the completely continuous form (2.13), has in accordance with the Hilbert alternative, a unique solution which converges in the space \( l_2 \).

$\S 3$. Let us now turn to the solution of (2.9) by the reduction method, retaining \( N \) terms:

$$x_q + \sum_{n=0}^{N} A_{qn} x_n = \beta_q b_q = B_q \quad (q = 0, 1, 2, \ldots, N). \quad (3.1)$$

According to the Cramer theorem,

$$x_j = \frac{\Delta_j}{\Delta} = \frac{1}{\Delta} \sum_{i=0}^{N} \beta_i \delta_{ij} \quad (j = 0, 1, 2, \ldots, N), \quad (3.2)$$

$$\Delta = \det Q_{q\theta_1} + A_{qn}\delta_{qn} = 0. \quad (3.3)$$

Here \( \Delta_j \) is the determinant of the matrix (3.3), in which the \( j \)-th column is replaced by a series of bounded elements \( \beta_0 \delta_{ij}, \beta_1 \delta_{ij}, \ldots, \beta_N \delta_{ij} \); \( \Delta_{ij} \) is the algebraic complement of the element \( \delta_{ij} \) of the matrix (3.3).

The expression for the velocity potential (2.8), with account taken of (3.2), now takes the form

$$\Phi(r, \theta, t) = \cdots \quad (3.4)$$