We examine hypersonic viscous gas flow past slender bodies. The entire disturbed flow region is divided into two regions; the viscous region where the boundary layer equations are valid, and the inviscid region located between the shock wave and the boundary layer, in which the flow is described by the ideal gas equations [1, 2].

We consider the case of strong interaction, when the parameter \( \chi = M^2(\eta_0)_{Re}^{1/2} \), where \( M \) is the Mach number and \( \eta_0 \) is the Reynolds number, is much greater than 1. In this region, for bodies with a power-law generatrix \( y = cx^n \) (the coordinates \( x \) and \( y \) are measured from the nose along and normal to the body) with \( n > 3/4 \), the interaction between the boundary layer and the outer flow predominates over the blunt effect [3]. This means that the pressure in the flow is determined primarily by boundary layer growth, while the body shape introduces only small perturbations. Thus, the pressure in the viscous region may be represented in the form of a series in which the principal term is determined by the pressure induced by the boundary layer on a flat plate, while the second and following terms account for the effect of the body shape and other factors.

We introduce immediately the following notation: \( u, u', v, v' \) are the projections of the velocity onto the \( x, y \)- and \( x', y' \)-axes, respectively, where \( x' \) and \( y' \) are measured from the nose along and normal to the generatrix, \( T, \rho, \rho, \mu, H, \) and \( \mu \) are the temperature, pressure, density, stagnation enthalpy, and viscosity of the gas; \( \psi \) is the stream function, \( N_{Re} \) is the Prandtl number. The subscripts ~, c, w, and e apply, respectively, to the free stream, shock wave, body surface, and edge of the boundary layer.

It was shown in [1, 2] that the equation of the shock wave which develops for hypersonic viscous flow past a flat plate will be \( \psi_0 \sim x^{3/4} \). For flow past two-dimensional and axisymmetric bodies with the generatrix \( y \sim x^{3/4} \), the principal term in the shock wave equation has the same form [3–5]. Therefore, it is natural to specify the shape of the shock wave for bodies \( \psi_0 = cx^n \) with the exponent \( 3/4 \leq n \leq 1 \) by the relation

\[
y_0 = ax^n \left( 1 + bx^{3/2} \right) \quad (0 \leq a \leq 1/\alpha).
\]

The constant \( \alpha \) may be found from the condition that the leading term of the expansion corresponds to flow past a flat plate; the constants \( b \) and \( \alpha \), associated with the body shape, will be determined by matching the solutions in the inviscid region and in the boundary layer.

Since the shock wave shape is given, the solution in the inviscid region may be completely defined. It will depend on the parameters \( a, b, \) and \( \alpha \).

On the basis of the method of plane sections, which is valid for hypersonic flow past slender bodies, we write the equations of motion as [6]

\[
\begin{align*}
\frac{\partial \rho}{\partial x} + \frac{\partial \rho u}{\partial y} &= 0, \\
\frac{\partial \rho u}{\partial x} + \frac{\partial \rho u^2}{\partial y} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial y}, \\
\frac{\partial \rho v}{\partial x} + \frac{\partial \rho uv}{\partial y} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial y},
\end{align*}
\]

The boundary conditions are specified at the shock wave front \( \psi_0 \):}

\[
\begin{align*}
\frac{\partial \rho}{\partial x} &= \frac{2}{\alpha + 1} \left( \frac{d\psi}{dx} \right)^2, \\
\frac{\rho_0}{\rho} &= x^{3/4}, \\
\frac{\psi_0}{\rho_0} &= \frac{2}{\alpha + 1} \frac{d\psi}{dx},
\end{align*}
\]

We introduce the new variable \( \xi = y/ax^{3/4} \) and seek \( \rho, \psi, \) and \( v_1 \) in the form

\[
\begin{align*}
\rho_1 &= \frac{\rho}{\rho_0} = x^{3/4} \left( \frac{\rho}{\rho_0} \right)^{1/3}, \\
v_1 &= \frac{v}{U_\infty} = \frac{3}{2} \frac{d\psi}{dx}, \\
\psi_1 &= \frac{\psi}{\psi_0} = \frac{3}{2} \frac{d\psi}{dx}.
\end{align*}
\]

The quantities \( \rho_1, \psi_1, \psi_1 \) satisfy the equations

\[
\begin{align*}
A \left( \frac{d\rho}{d\xi} \right) - \frac{\partial \rho}{\partial \xi} &= \frac{\partial \rho}{\partial \xi}, \\
A \left( \frac{d\rho}{d\xi} \right) + \frac{\partial \rho}{\partial \xi} &= \frac{\partial \rho}{\partial \xi}, \\
A \left( \frac{d\psi}{d\xi} \right) - \frac{\partial \psi}{\partial \xi} &= \frac{\partial \psi}{\partial \xi}, \\
A \left( \frac{d\psi}{d\xi} \right) + \frac{\partial \psi}{\partial \xi} &= \frac{\partial \psi}{\partial \xi}, \\
A \left( \frac{d\psi}{d\xi} \right) - \frac{\partial \psi}{\partial \xi} &= \frac{\partial \psi}{\partial \xi}.
\end{align*}
\]

Equations (5) and (6) were solved numerically in the region from the shock wave to the streamline \( \psi = 0 \), which corresponds to \( \xi = \psi/ax^{3/4} \). As shown in [1–3], the outer edge of the boundary layer is nearly a streamline. Therefore, to match the solutions in the inviscid region and in the boundary layer we can consider that the streamline \( \psi = 0 \)
in the first of these regions corresponds to the edge of the boundary layer, i.e.,
\[\frac{dy_e}{dx} = \frac{v_e(x, y)}{U_e}, \quad y_e = 0 \quad \text{for} \quad x = 0. \quad (7)\]

This equation yields the dependence of \(y_e\) on the parameters of the inviscid region. On the other hand, \(y_e\) may be obtained from the solution of the boundary layer equations. Equating these two expressions, we obtain the relation connecting the constants \(b\) and \(a\) with the body shape.

Writing \(y_e\) in the form of a series and retaining only two terms, we obtain from (7)
\[y_e = \frac{2}{\pi + \frac{1}{4}} \left(1 + \frac{V_1}{B} \frac{dV_e}{dz}ight), \quad B = (1 + 4k) V_e \frac{dV_e}{dz}. \quad (8)\]

Thus, the pressure and normal velocity component at the edge of the boundary layer are
\[p_e = \frac{9}{8(k + 1)} M^2 \rho U_e d_e x^2 + \left(1 + \frac{V_1}{B} \frac{dV_e}{dz} + \frac{\rho}{\rho_e} \right) b z^{3/4}, \quad (9)\]

where \(V_1, V_e, dV_e/dz\), and \(d\rho_e/dz\) are determined for \(x = \pm h_e\).

Let us turn to the study of the boundary layer.
The boundary layer equations are written in the variables \(x'\) and \(y'\), which are connected with \(x\) and \(y\) by the relations
\[x' = \int \left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right] dx, \quad y' \cos \left(\arg \left(\frac{\partial y}{\partial x}\right)\right) = y - y_e. \quad (10)\]

The derivative \(\partial y/\partial x < \partial y_e/\partial x = \pm x^{3/4}\); therefore, in the region \(x^{3/4} \ll 1\) the term \((\partial y_e/\partial x)^2\) may be neglected in comparison with 1 and then \(x' \approx x\). Similarly
\[y' = y - y_e. \quad (11)\]

The boundary layer equation has the form [3]
\[\frac{\partial p}{\partial x} - \frac{\partial v}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \quad p = \rho H, \quad \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{1}{N_T} \frac{\partial \rho}{\partial y} \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial v}{\partial y}\right). \quad (12)\]

Let us consider the case in which \(P = 1\), the viscosity is proportional to the temperature, and the body is thermally insulated. In this case we have the familiar energy integral
\[\frac{\partial \rho}{\partial \rho} = \frac{1}{2} \left(1 + \frac{3}{3} \frac{\partial u}{\partial x} \right), \quad \rho = \frac{1}{2} M^2 \left[1 + \frac{3}{2} \frac{\partial u}{\partial x}\right]. \quad (13)\]

Comparing \(y'_e\) and \(y_e\) using (11) and bearing in mind that the leading terms of the expansion correspond to flow past a flat plate, we obtain the expressions for \(a\) and \(y_w\):
\[\frac{\partial f}{\partial \eta} + f \frac{\partial f}{\partial \eta} - \frac{1}{2} \ln \left[1 - \left(\frac{\partial f}{\partial \eta}\right)^2\right] = \frac{2}{\eta} \left(\frac{\partial f}{\partial \eta} \frac{\partial f}{\partial \eta} - \frac{\partial f}{\partial f}\right). \quad (14)\]

We have already noted that the pressure in the boundary layer equals the pressure in the inviscid region for \(\psi = 0\). Since \(x' = x\), it follows that \(p(x') = p(x)\). We need to determine the connection between \(x'\) and \(t\). Using the energy integral and bearing in mind that for a thermally insulated body \(\rho_{\text{es}}\) equals the stagnation flow viscosity, we obtain
\[t = \frac{\rho_{\text{es}}}{\rho} \left(1 + \frac{\psi}{2a + 1}\right), \quad \beta = \frac{9}{4} \frac{x}{x + 1} \frac{U_e}{\rho_e} \rho \frac{dU_e}{dz} \frac{dU_e}{dz} \frac{dU_e}{dz}, \quad (15)\]

As a result
\[\frac{d \ln p}{d \ln t} = \left[1 + \frac{4a(a + 1) \rho_{\text{es}}}{(2a + 1) \rho_{\text{es}}}ight], \quad (16)\]

Therefore, it is natural to seek \(f\) in the form
\[f = f_s(\eta) + 2 \rho_{\text{es}} \rho_{\text{en}}(\eta). \quad (17)\]

Substituting the expansion of \(d \ln p / d \ln t\) and \(f\) into (14), we obtain the equations for \(f_s\) and \(f_1\):
\[\frac{d f_s}{d \eta} + f_s \frac{d f_s}{d \eta} - f_1 \left[1 - \frac{\partial f_s}{\partial \eta}\right] = 0, \quad \frac{d f_1}{d \eta} + f_1 \frac{d f_1}{d \eta} - \frac{1}{\rho_e} \left[1 - \frac{\partial f_1}{\partial \eta}\right] + \frac{\partial f_1}{\partial \eta} = 0, \quad (18)\]

with the boundary conditions
\[f_s = f_1 = \frac{d f_s}{d \eta} = \frac{d f_1}{d \eta} = 0 \quad \text{for} \quad \eta = 0, \quad \frac{d f_s}{d \eta} = 1, \quad \frac{d f_1}{d \eta} = 0 \quad \text{for} \quad \eta = \infty. \quad (19)\]

These equations were solved numerically.

From (12), using the energy integral, we obtain the boundary layer thickness:
\[y'_e = \frac{2}{3} \left(1 - \frac{1}{\eta} \right) \frac{\partial M}{\partial x} \left(\frac{\rho}{\rho_e} \frac{d U_e}{d \eta}\right) \frac{1}{\beta} \frac{d \eta}{\beta}, \quad (20)\]

Comparing \(y'_e\) and \(y_e\) using (11) and bearing in mind that the leading terms of the expansion correspond to flow past a flat plate, we obtain the expressions for \(a\) and \(y_w\):
\[a^2 = \frac{\rho}{\rho_e} \left(1 - \frac{1}{\eta} \right) \frac{\partial M}{\partial x} \left(\frac{\rho}{\rho_e} \frac{d U_e}{d \eta}\right) \frac{1}{\beta} \frac{1}{\rho_e} \frac{d \eta}{\beta} \frac{d \eta}{\beta}, \quad (21)\]