1. INTRODUCTION

Theoretical investigation of diffraction of electromagnetic waves leads to boundary-value problems of electrodynamics or to the corresponding integral equations. It is well known that, as a result of the mathematical difficulties, diffraction theory tends to prefer configurationally simple entities [1]. At the same time, intensive developments in computer resources provide more and more justification for the widespread tendency to introduce mathematical models into radio engineering. In designing antenna, waveguide, and other devices, there is increasing emphasis on methods that are suitable for creating mathematical models that are close to reality.

Prominent in this respect is the role of variational methods in electrodynamics; they are becoming more firmly established as computers become more sophisticated. The initial idea here is that an electrodynamic problem is associated with some functional and its steady-state value is sought on a set of "permissible" functions; it turns out that the sequence of functions that realize the approximation to the steady-state value tends (in some metric) to the solution of the problem. What may be of great practical importance here is not the realizing function but the steady-state value of the functional itself, which yields some characteristic parameter of the problem. This approach (or, as is said, solution of a variational problem) is fairly flexible, since to a certain (and sometimes considerable) extent the configurational and analytical constraints are eliminated. It is interesting that, in simple cases, it may be useful to substitute an intuitively determined function (obtained from physical considerations) into the functional. As for more or less complicated electrodynamical problems, in particular, diffraction problems, the application of variational methods was almost fruitless until large computers made their appearance.

Now we should note that the algorithmization of variational methods for the problems under consideration usually involves the projection from functional space onto a finite-dimensional subspace. This is the essence of the familiar Ritz method, and hence this method is in the category of projection methods.

Although an exposition of the fundamentals of variational calculus and variational methods in electrodynamics [2, 3] is not a part of this survey, we will require a brief discussion of some facts for what follows.

Let us formulate two problems:

\[ \hat{L}v = f \quad \text{and} \quad \tilde{L}u = g, \]  

where \( \hat{L} \) is a linear operator (e.g., a boundary-value problem operator or an integral operator); \( \tilde{L} \) is an adjoint operator; and \( f \) and \( g \) are independent specified "excitation functions." We associate the following functional with these formulations:

\[ F(u, v) = (Lu, v) - (f, v) - (u, g) \]  

(1.2)

(The scalar products are written on the right).

The Ritz method for problems (1.1) is based on a selection of some basis \( \{u_n\} \) in functional space. We set up the representations

\[ u^N = \sum_{n=1}^{N} a_n u_n \quad \text{and} \quad v^N = \sum_{n=1}^{N} b_n u_n \]  

(1.3)

(\(a_n\) and \(b_n\) being indeterminate coefficients). In the simplest case in which the basis functions belong to the domain of definition of operators \(L\), \(\mathcal{L}(u_n \in D_L, \tau)\), we can validly employ the ordinary substitution \(u \rightarrow u^N\), \(v \rightarrow v^N\), which brings functional \(F(u, v)\) to the form

\[
F^N = \sum_{k, s=1}^{N} a_s b_s^* (L u_k, u_k) - \sum_{s=1}^{N} |\beta_s f |^2 + a_s (u_s, g)|.
\]

(1.4)

Obviously, \(F^N\) is a function of variables \(a_1, a_2, \ldots, a_N, b_1^*, b_2^*, \ldots, b_N^*\). Formally, the Ritz method as applied to the first problem in (1.1) involves the computation and vanishing of the partial derivatives of \(F^N\) with respect to all \(b_n^*:\)

\[
\frac{\partial F^N}{\partial \beta_n^*} = 0 \quad (n = 1, 2, \ldots, N).
\]

(1.5)

As we can see from (1.4), this operation leads to the following "algebraic image" of problem \(Lu = f:\)

\[
La = f.
\]

(1.6)

This is an inhomogeneous system of linear equations in \(a\) [a vector made up of coefficients \(a_n\) of sum \(u^N\) in (1.3)]; \(L\) is a matrix with elements \(L_{kn} = (L u_k, u_k)\); and \(f\) is a vector with components \(f_k = \{f, u_k\}\).

Note that when \(L\) is self-adjoint (\(\mathcal{L} = L\)), algebraic form (1.6) can be obtained by the Ritz method from the quadratic functional \(F(u) = (Lu, u) - 2\{f, u\}\), as is done to substantiate the method [2].

The projection nature of system (1.6) is so obvious that there is no need to resort to any functionals to obtain it. Starting directly from the first formulation (1.1), we allow for the fact that the null element of functional space \(Lu = f\) is orthogonal to all \(u_n\). Using the representation of \(u^N\) in (1.3) and imposing the requirement of orthogonality up to \(N\) inclusive, we have, after the substitution \(u \rightarrow u^N\),

\[
(L u^N - f, u_k) = 0. \quad (k = 1, 2, \ldots, N).
\]

(1.7)

This yields system (1.6) directly. As we know, process (1.7) is called the Bubnov–Galerkin method.

Algorithmically, of course, the manner of reasoning that leads from initial problem \(Lu = f\) (1.1) to algebraic problem \(La = f\) (1.6) makes absolutely no difference. The direct projection approach embodied in the Bubnov–Galerkin method is itself devoid of variational notions (although this method is traditionally grouped with variational methods [2]). The realization of the algorithm is based on the application of computational methods of linear algebra to system (1.6).

There remains, however, the problem of substantiating the algorithm, i.e., of at least proving that \(u^N \rightarrow u\) converges in functional space as \(N \rightarrow \infty\). This proof can employ variational principles or be entirely free of them. But it must be borne in mind that formal substantiation still does not guarantee success in realizing the algorithm; convergence can be unacceptably slow, or the stability problem may require particular attention.

The more problematic it is to obtain a satisfactory representation of \(u^N\) by solving system (1.6) of high order \(N\), the more important it is to be able to evaluate the realized degree of accuracy of \(u^N\) or the characteristic parameters expressed as functionals \(\psi(u^N)\). The development of variational methods and allied principles makes it possible to approach the problem of setting up a priori estimates for the computational error for such characteristic parameters. A posteriori estimates can provide the basis for the elaboration of nonalgebraic algorithms for diffraction problems with a certain error margin.

Reduction of problem \(Lu = f\) (1.1) to system of algebraic equations \(La = f\) (1.6) yields a general representation of one class of variational (projection) algorithms for diffraction problems. The specifics are determined by the form of operator \(L\) and the choice of basis \(\{u_n\}\). It is typical to employ functions \(u_n\) that do not belong to domain \(D_L\); direct substitution of \(u^N\) into (1.2) or (1.7) is impossible in this case; the scalar products are transformed with integration by parts [2, 3].

Another important class of algorithms is generated by using representations \(u^N\) and \(v^N\) with coefficients that depend on one spatial coordinate; \(a_n = a_n(t)\), \(b_n = b_n(t)\). Unlike the preceding case, problem (1.1) reduces here to a system of ordinary differential equations. As before, Bubnov–Galerkin and Ritz processes lead to the same results. The method was presented in detail in [4], without reference to diffraction problems.

The Trefftz method, better known in electrodynamics as the method of partial regions, is also of great importance. It is customarily employed when there are no volumetric sources \(f = 0\) in (1.1). A feature of the method is that it employs basis functions \(u_n\) that are governed by the basic equation of the problem (i.e., \(Lu_n = 0\)) but do not satisfy some (or all) boundary conditions. The system of basis functions should be complete only