1. Introduction

In the construction of unified models of elementary-particle interaction the original symmetry group must be spontaneously broken. This is generally done by a mechanism based on the introduction of one or several multiplets of scalar Higgs fields. In accordance with the general principles of the theory, the Higgs potential must satisfy conditions that ensure it is renormalizable and bounded below, i.e., it must have the form of a polynomial of not higher than fourth degree and be positive at large values of the fields.

In the simplest cases with a small number of coupling constants (an example is the Salam-Weinberg model with one Higgs doublet), it is not difficult to find the only values for which the Higgs potential is bounded below. In models of the weak and electromagnetic interactions containing several scalar doublets, and also in grand unification theories based on groups of higher rank and having several multiplets of scalar particles, the Higgs potentials have much more complicated structures. As a result, the problem of finding the necessary and sufficient conditions on the parameters of the potential for which it is bounded below becomes nontrivial.

It should be noted that in a number of studies [1-5] and for some potentials only sufficient conditions have been found; these far from exhaust the complete region of parameters in which the requirement of positivity at large values of the scalar fields is satisfied.

In the present paper, we consider, in the tree approximation, a number of Higgs potentials that are interesting from the point of view of physical applications and obtain necessary and sufficient conditions for them to be bounded below. We also require this property to hold for all values of the dimensional constants of the coupling of the scalar particles with one another, this being equivalent to positive definiteness of the part of the potential that contains only fourth powers of the fields.

From the methodological point of view, the problem of necessary and sufficient conditions for boundedness below is a generalization of the problem of finding residual invariance groups in the presence of spontaneous symmetry breaking* (see, for example, [5-8]). The point is that all information about possible symmetry types of the theory can be obtained by studying the invariant properties of the point of

* This applies only to potentials that do not contain terms of the type $\varphi^3$.

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Necessary and sufficient conditions are obtained for Higgs potentials to be bounded below when they are constructed from: 1) two doublets, and also two doublets and a singlet of SU(2). 2) the adjoint and vector representations of SO(n). For a potential constructed from the adjoint and fundamental multiplets of SU(n), the problem of necessary and sufficient conditions has been only partly solved.
minimum of the potential (denoted in all sections of this paper by $V_1$). On the other hand, it is precisely
knowledge of the smallest value of $V_1$ that plays the decisive role in finding the necessary and sufficient
conditions for positivity of the Higgs potential.

In Sec. 2, taking the example of the Salam-Weinberg model with two doublets, we explain the
method used to obtain the estimates. These results are used in Sec. 3, in which we investigate a Higgs
potential constructed from two doublets and a singlet of the group SU(2) (such a structure of the scalar
sector can also be used in SU(2) × U(1) × U(1) theories, as noted in [2], to suppress monopole production).
In grand unification models based on the groups SO(n), one of the spontaneous symmetry breaking chains
contains the adjoint and the vector multiplet of scalar fields. The Higgs polynomial of these representations
of SO(n) is considered in Sec. 4. In the fifth and final section, we give some necessary and sufficient
conditions for a SU(n) invariant potential formed by the adjoint and fundamental representations to be
bounded below. We have succeeded in obtaining bounds only for the case when the coupling constants $\alpha$ and
$\beta$ (see Eqs. (21) and (20)) have the same signs. If $\alpha$ and $\beta$ have opposite signs, the methods proposed in
this paper yield only sufficient conditions.

2. Salam-Weinberg Model with Two Doublets

In this section, we consider the Higgs potential in the Salam-Weinberg model with two scalar
doublets $\phi_1$ and $\phi_2$, the homogeneous part of fourth order having the form $V = V_0 + V_1$, where
\begin{align*}
V_0 &= \lambda_1 (\phi_1^* \phi_1) + \lambda_2 (\phi_2^* \phi_2)^2 + 2\gamma (\phi_1 \phi_2^*) (\phi_2 \phi_1^*), \\
V_1 &= \rho (\phi_1^* \phi_2) (\phi_2^* \phi_1) + \eta (\phi_1^* \phi_1) + 2\eta^* (\phi_2^* \phi_2)^3
\end{align*}

(other combinations of $\phi_1$ and $\phi_2$ must not occur in the expression for $V_1$, since invariance with respect to
sign reversal of the fields is assumed). Given that the Higgs potential is bounded below, the requirement $V \geq 0$ is obviously necessary for all $\phi_1$ and $\phi_2$. And then along the ray on which $V = 0$ the relationships
between the dimensional parameters, including the coefficients of the powers of the fields less than the
fourth, begin to play a decisive role. Therefore, to simplify the problem we shall, here and in the following
sections, assume that the potential is bounded below for all values of the dimensional constants. Necessary
and sufficient for this is $V > 0$ for all $\phi_1$ and $\phi_2$.

Suppose $V(\Phi_1, \Phi_2) = \min V(q_1, q_2)$ subject to the condition
\begin{equation}
\phi_1^* \phi_1 = \Phi_1, \quad \phi_2^* \phi_2 = \Phi_2.
\end{equation}

Then $V(q_1, q_2) > 0$ if and only if $V(\Phi_1, \Phi_2) > 0$ for all $\Phi_1, \Phi_2 > 0$ that do not vanish simultaneously. It is
obvious that $V_0$ depends only on $\Phi_1$ and $\Phi_2$; therefore, to find $V(\Phi_1, \Phi_2)$ it is sufficient to minimize $V_1(q_1, q_2)$
under the condition (3). Since the model has SU(2) invariance, without loss of generality we can set
$q_1^+ = (0, \sqrt{\Phi_1})$. We introduce the new notation $q_1^+ = (x_1, x_2)$, $\text{Im} x_1 = \sin \omega \sqrt{\Phi_1 - |x_1|^2}$, $\text{Re} x_2 = \cos \omega \sqrt{\Phi_1 - |x_1|^2}$. Then $V_1$ takes the form
\begin{equation}
V_1(\Phi_1, \Phi_2) = \left\{ \begin{array}{ll}
0 & \text{for } p - 2|\eta| > 0, \\
\Phi_1 (\Phi_2(p - 2|\eta|)) & \text{for } p - 2|\eta| < 0.
\end{array} \right.
\end{equation}

where $\tan \delta = \frac{\text{Im} \eta}{\text{Re} \eta}$. The smallest value of the function (4) with respect to the variables $\omega$ and $|x_1|$, which lie in the intervals $0 \leq \omega \leq 2\pi$, $0 \leq |x_1| \leq \Phi_1$, is
\begin{equation}
P_1(\Phi_1, \Phi_2) = \min P_1 = \left\{ \begin{array}{ll}
0 & \text{for } p - 2|\eta| > 0, \\
\Phi_1 (\Phi_2(p - 2|\eta|)) & \text{for } p - 2|\eta| < 0.
\end{array} \right.
\end{equation}

Thus, we have obtained the function $V(\Phi_1, \Phi_2) = V_1(\Phi_1, \Phi_2) + V_0(\Phi_1, \Phi_2)$. If $\phi_1 = 0$, then $V(\Phi_1, \Phi_2) > 0$ for all $\Phi_2 > 0$ in the case $\lambda_2 > 0$. Suppose $\phi_1 \neq 0$. The function $V(\Phi_1, \Phi_2)$ can be represented in the form
\begin{equation}
V(\Phi_1, \Phi_2) = \left\{ \begin{array}{ll}
\Phi_1 (\lambda_1 + \lambda_2 \Phi_2^2 + 2\gamma) & \text{for } p - 2|\eta| > 0, \\
\Phi_1 (\lambda_1 + \lambda_2 \Phi_2^2 + h(2\gamma + p - 2|\eta|)) & \text{for } p - 2|\eta| < 0,
\end{array} \right.
\end{equation}

where $0 \leq h = \Phi_2 / \Phi_1 < \infty$. It follows from the lemma in Appendix 1 that the function (6) will be positive with
respect to the variable $h$ provided
\begin{equation}
\{ \lambda_1, \lambda_2 > 0; \gamma > -\sqrt{\lambda_1 \lambda_2}; p > 2|\eta| \} \cup \{ \lambda_1, \lambda_2 > 0; 2\gamma + p - 2|\eta| > -2\sqrt{\lambda_1 \lambda_2}; p < 2|\eta| \}.
\end{equation}

The set (7) is the necessary and sufficient condition for the Higgs potential in the Salam-Weinberg model
with two doublets to be bounded below.