SPATIALLY DISTRIBUTED CLASSICAL LAGRANGIAN MECHANICS

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It is well known that the existence of two nontrivial integrals of the motion makes it possible to parametrize the motion of a Lagrangian rigid body by two variables. On the basis of this fact it is shown that certain combinations of the quantities that characterize the trajectory of such a body satisfy well-known nonlinear equations: sine—Gordon, Korteweg—de Vries, Klein—Gordon, and nonlinear Schrödinger equation.

Numerous investigations of special cases of the motion of rigid bodies [1—5] have established an intimate connection between them and the equations that describe a wide variety of physical phenomena, for example, the Landau—Lifshitz equation, the equation of motion of a charged body in an electromagnetic field, the evolution equation of a two-atom in an electromagnetic field. In their turn, study of the properties of these phenomena has shown [5—8] that their analytic description is concentrated around a well-known group of nonlinear evolution equations: Korteweg—de Vries (KdV), Klein—Gordon (KG), sine—Gordon (sG), nonlinear Schrödinger equation (NSE). As a natural continuation of these investigations, there is interest in the problem of establishing the direct connection between the nonlinear equations and the equations that describe the motion of rigid bodies.

It is well known that the motion of Lagrangian rigid bodies can be characterized by two variables. On the basis of this fact, it is shown in the present paper that specially chosen combinations of quantities that characterize the rigid-body motion satisfy one of the equations of the following set: NSE, KdV, sG, KG.

A heavy rigid body with one fixed point is described in the Lagrange—Poisson case by the system of equations [9]

\[
\begin{align*}
\frac{dp}{dt} &= rq - v, \\
\frac{dq}{dt} &= -rp + u, \\
\frac{dr}{dt} &= 0; \\
\frac{du}{dt} &= rv - qw, \\
\frac{dv}{dt} &= -ru + pw, \\
\frac{dw}{dt} &= uq - vp;
\end{align*}
\]

which has four integrals of the motion:

\[
p^2 + q^2 - 2w = E, \quad up + qv = K, \quad u^2 + v^2 + w^2 = R^2, \quad r = \text{const.}
\]

Here, the evolution parameters have the following meaning [9]: \(\{p, q, r\} = F\) are the components of the angular velocity, \(\{u, v, w\} = M\) are quantities proportional to the components of a fixed vertical vector in a coordinate system attached to the body [for the form of expression of (1), they have the dimensions of angular velocity]. The integrals \(E, K, R, r\) are known functions in the six-dimensional parameter space \(R^6(F, M) = R^3(F) \times R^3(M)\); \(E\) is the total energy, \(K\) is the scalar product of the kinetic angular momentum and the vertical vector, and \(R\) is the length of the vertical vector.

It is shown in [10] that in the variables \(p, q, w, \varphi (\varphi = \tan^{-1} v/u)\), which are symplectic coordinates, the system (1) can be written in the canonical form

\[
\frac{dx^i}{dt} = \{x^i, H\} = \sum_{k,l} (z^k, z^l) \frac{\partial x^i}{\partial z^k} \frac{\partial H}{\partial z^l}, \quad \{z^k, z^l\} = \Omega^{kl}, \quad \Omega = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix};
\]

where \(\{x^i, H\}\) is the Poisson bracket, \(x^1 = p, x^2 = q, x^3 = w, x^4 = \varphi\); \(\Omega\) is a structure matrix, \(E\) is the unit 2×2 matrix, \(H\) is the Hamilton function formed from the integrals of the motion (2):

\[
H = \frac{r}{2}(p^2 + q^2 - 2w) - (pu + qv).
\]

Equations (3) can be written out more fully as

\[
\begin{align*}
\frac{dp}{dt} &= rq - \sqrt{R^2 - w^2} \sin \varphi, \\
\frac{dw}{dt} &= \sqrt{R^2 - w^2} (q \cos \varphi - p \sin \varphi),
\end{align*}
\]
\[
\frac{dq}{dt} = -rp + \sqrt{R^2 - w^2} \cos \varphi, \quad \frac{d\varphi}{dt} = -r + \frac{w}{\sqrt{R^2 - w^2}(p \cos \varphi + q \sin \varphi)}; \quad (5)
\]

\[E = p^2 + q^2 - 2w, \quad K = \sqrt{R^2 - w^2}(p \cos \varphi + q \sin \varphi), \quad H = \frac{r}{2}E - K, \quad R, r - \text{const.}\]

Here, the variables \(p, q, w, \varphi\) are canonically conjugate:

\[\{p, q\} = -1, \quad \{w, \varphi\} = -1\]

and specify the generic point of the four-dimensional phase space \(N\). In [10], the classical system (5) with the phase space \(N\) was defined in accordance with the terminology of [11-12] as a Lagrangian classical mechanics \(L\).

Usually, the description of the rigid-body motion (1) or (5) is restricted to specifying the dependence of the evolving parameters \(\{p, q, r\}, \{u, v, w\}\) on a single variable — the time \(t\). However, besides the energy integral \(E\) the system possesses the additional nontrivial integral \(K\), which is in involution with \(E\) with respect to the Poisson bracket (3). Then by Liouville's theorem [13] a joint level surface \(M_2\) of the functions \(E\) and \(K\) is a smooth two-dimensional submanifold in \(N\) that is invariant with respect to the Hamiltonian vector fields (Hamiltonian flows) generated by the functions \(E\) and \(K\) and is diffeomorphic to the Liouville torus \(T^2\). The generic point of the Liouville torus can be parametrized both by the time of the system with the integral \(E\) and the time of the system generated by the integral \(K\). Therefore, the evolution parameters \(\{p, q, r\}, \{u, v, w\}\) become quantities parametrized by two variables: \(t_1\) and \(t_2\). To specify the dependence of the evolution parameters with respect to these variables, we use the fact that each of the integrals \(E\) and \(K\) that generate the corresponding flows is also an integral for the other Hamiltonian flow and that these integrals mutually commute. Then the evolution of the variables that characterize the motion under the influence of the flows with the times \(t_1, t_2 = t\) is determined [14,15] by the Hamiltonian systems

\[
\frac{dz^i}{dt} = \{z^i, H_\alpha\} = \sum_{\alpha, \beta} \Omega^{i\beta}_{\alpha} \frac{\partial H_\alpha}{\partial z^\beta}; \quad \alpha = 1, 2. \quad (6)
\]

In order to make the systems (6) consistent with the equations of motion (5), we have chosen as Hamiltonians \(H_\alpha\), not the integrals \(E\) and \(K\) themselves, but linear combinations of them:

\[H_1 = H = \frac{r}{2}(p^2 + q^2 - 2w) - (pu + qv), \quad H_2 = h = 2r(pu + qv).\]

The dimension of the integral \(h\) is chosen in such a way that the dimension of the parameter \(\xi = t_2\) is equal to the dimension of the spatial coordinate.

Written out in full, Eqs. (6) become

\[
\begin{align*}
pt &= rq - \sqrt{R^2 - w^2} \sin \varphi, & pq &= 2r\sqrt{R^2 - w^2} \sin \varphi, \\
qt &= -rp + \sqrt{R^2 - w^2} \cos \varphi, & q\varphi &= -2r\sqrt{R^2 - w^2} \cos \varphi, \\
w_\xi &= \sqrt{R^2 - w^2}(q \cos \varphi - p \sin \varphi), & w_\xi &= -2r\sqrt{R^2 - w^2}(q \cos \varphi - p \sin \varphi), \\
\varphi_\xi &= -r + \frac{w}{\sqrt{R^2 - w^2}(p \cos \varphi + q \sin \varphi)}, & \varphi_\xi &= -\frac{2rw}{\sqrt{R^2 - w^2}(p \cos \varphi + q \sin \varphi)}. \quad (8)
\end{align*}
\]

We shall call the Lagrangian classical mechanics \(L\) with the Hamiltonian systems (6) the spatially distributed classical mechanics \(L_s\).

On the basis of the system of equations (7) and (8), we show that \(\{p, q, r\}, \{u, v, w\}\) satisfy nonlinear evolution equations.

We note first that the parameters \(u\) and \(v\) can be expressed in terms of the canonical variables:

\[u = \sqrt{R^2 - w^2} \cos \varphi, \quad v = \sqrt{R^2 - w^2} \sin \varphi.\]

Then, augmenting the systems (5) and (6) with the equations for \(u\) and \(v\), we obtain

\[
\begin{align*}
p_\xi &= 2ru, & u_\xi &= 2rw, \\
q_\xi &= -2ru, & v_\xi &= -2rpw, \quad (9)
\end{align*}
\]

For \(p\) and \(q\), direct calculations by means of (9) lead to the nonlinear Schrödinger equations