ASPECTS OF THE STUDY OF MODEL HAMILTONIANS OF
STATISTICAL PHYSICS

1. BASIC THEOREMS FOR THE FREE ENERGY

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The methods developed by N. N. Bogolyubov, Jr. to study model systems of statistical physics with four-fermion interaction are used to study a system with a Hamiltonian of general form in which the dynamical operators are not particularized but one requires the fulfillment of certain very general conditions. Special cases of this system are the BCS model in the theory of superfluidity and models of ferromagnets of Heisenberg and Ising type. Theorems are proposed for the free energy of such a model system, these generalizing the known theorems [4, 5].

Asymptotically exact methods for investigating many-particle systems were developed by Bogolyubov in his pioneering work [1, 2] in the creation of a microscopic theory of superfluidity. Considerable attention was devoted in these papers to a mathematically rigorous justification of the proposed methods of investigation [3]. Among the model problems of modern statistical physics a particular position is occupied by those that allow an exact solution in the sense of the thermodynamic limit (\( V \to \infty \), \( N \to \infty \), \( N/V = \text{const} \), where \( V \) is the volume of the system and \( N \) is the number of particles). Subsequently, for such model systems N. N. Bogolyubov, Jr. proposed methods of a mathematically rigorous proof of the asymptotic exactness of the results obtained [4] and constructed [5] a general method for studying the asymptotic behavior of such quantities as the free energy (and its derivatives) and the equal-time and many-time correlation functions and Green's functions.

These methods, which were originally developed to study the Bardeen–Bogolyubov model systems [1–5] with Hamiltonian of the form*

\[
H = \sum_{\alpha} T \alpha_{-\alpha} \frac{1}{2V} \sum_{\beta} l(\alpha, \beta) \alpha_{\beta} \alpha_{-\beta},
\]

were successfully used in investigations in quasispin models [6–7] with a Hamiltonian of Thirring type:

\[
H = \sum_{\mu=1}^{2T} \sum_{\mu'=1}^{2T} \sigma_{\mu} \sigma_{\mu'},
\]

and in the Ising model [8–9]:

*The following notation is introduced in (1)–(4): \( f = (p, s) \), \(-f = (-p, s)\), where \( p \) is the momentum and \( s \) the spin index; \( T_f = p^2/(2m) - \mu \), where \( \mu \) is the chemical potential; \( l(f, f') \) is the interaction kernel, which has definite symmetry properties; \( \alpha_f \) and \( \alpha_{f}^+ \) are Fermi operators. The positive quantity \( T_c \) is the interaction constant (in appropriate units, equal to the critical temperature); \( \sigma_p, \sigma_{p}^+, \sigma_l^z \) are operators of Pauli type; \( J \) is the number of pairing states; \( I \) is the constant of the exchange interaction; \( \mu_0 \) is the magnetic moment of the particles; \( N \) is the number of particles; \( \hbar \) is the external magnetic field.

Subsequently, it was shown that for model problems with Hamiltonians of slightly more general form, for
equation, in the case of a system of superfluid type \( H = \sum_{p,s,s'} t (p) \alpha_{p,s} \alpha_{p',s'} \),
the method of [1-5] enables one to find asymptotically exact expressions for the equal-time and many-time
correlation functions and Green's functions.

Undoubted interest attaches to a further development of mathematical methods for investigating model
problems with Hamiltonian of the general form
\[
H = T - 2V \sum_{a=1}^{l} \sum_{a_{1},a_{2} \in \mathbb{Z}} g_{a} J_{a_{1}} J_{a_{2}}^{+},
\]
in which the dynamical operators \( T \) and \( J_{a} \) are not particularized, but one requires fulfillment of the condi-
tions
\[
T = T^{+}, \quad \| J_{a} \| \leq A_{1}, \quad \| TJ_{a} - J_{a} T \| \leq A_{1},
\]
where \( V \) is the finite volume of the system, \( A_{1} \) are certain positive constants as \( V \to \infty \) \((N/V = \text{const})\). The
coefficients \( g_{a} \) in (5) are chosen positive because we here consider only systems with negative interaction.

As "approximating" system to the model with the Hamiltonian (5) we propose a system described by
the Hamiltonian
\[
H_{a}(C) = T - 2V \sum_{a=1}^{l} \sum_{a_{1},a_{2} \in \mathbb{Z}} g_{a} (C_{a} J_{a_{1}}^{+} J_{a_{2}} + J_{a_{1}}^{+} J_{a_{2}}),
\]
which depends on the parameter \( C \), which is a vector \( C = (C_{1}, C_{2}, \ldots, C_{l}) \) in the \( l \)-dimensional complex
space \( E_{l} \); the constant \( \mathcal{X}(c) \) is introduced in formula (7) for reasons of convenience and, when needed,
can serve as a normalization parameter. It should be noted that for the Hamiltonian (7) the free energy
per unit volume\(^{\dagger}\) is a function of the parameter \( C \), defined in the space \( E_{l} \) of all points \( C \). In what follows,
the expression \( \min f(C) \) means everywhere the absolute minimum of the function \( f(C) \) in \( E_{l} \). Note that the
procedure for minimizing the free energy per unit volume for the Hamiltonian (7) as a function of the
parameter \( C \) enables one to obtain selfconsistency conditions [5] that determine the set of points \( \bar{C} = (\bar{C}_{1}, \bar{C}_{2}, \ldots, \bar{C}_{l}) \) at which the absolute minimum of \( f_{v}(H_{a}) \) is attained, i.e.
\[
f_{v}(H_{a}(\bar{C})) = \min_{(c)} f_{v}(H_{a}(C)).
\]

In this paper, requiring fulfillment of the conditions (6), we show that the expression for the free energy
per unit volume for the Hamiltonian (5) is asymptotically close to the corresponding expression
for the free energy of the "approximating system" with the Hamiltonian (7).

For our investigation it is more convenient to consider the so-called Hamiltonian with sources:
\[
h = T - 2V \sum_{a=1}^{l} \sum_{a_{1},a_{2} \in \mathbb{Z}} (v_{a} J_{a_{1}}^{+} J_{a_{2}} + v_{a}^{*} J_{a_{1}}),
\]
in which the parameters for including the sources, \( v_{a} \) and \( v_{a}^{*} \), are chosen proportional to \( \bar{C}_{a} \) and \( \bar{C}_{a}^{*} \),
respectively, with positive coefficients of proportionality:
\[
v_{a} = r_{a} \bar{C}_{a}, \quad v_{a}^{*} = r_{a} \bar{C}_{a}^{*}, \quad r_{a} > 0, \quad a = 1, 2, \ldots, l.
\]
As is shown in [5], the choice of these parameters in the form (10) enables one subsequently to avoid dif-
ficulties in the correct definition of quasiaverages for the considered model systems.

\(^{\dagger}\)The free energy per unit volume for a system with arbitrary Hamiltonian \( H \) we mean \( f_{V} = -\frac{1}{V} \ln \text{Sp} e^{-H/\theta} \), where the parameter \( \theta \) is the temperature expressed in energy units.