SPECTRAL-SOURCE-LIKE EXPANSIONS IN THE THEORY
OF WAVE PROPAGATION AND THE QUANTUM THEORY
OF POTENTIAL SCATTERING

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Linear nonself-adjoint differential operators are used to construct and investigate spectral-source-like solutions (expansions in normal waves) of the problems of the theory of radio wave propagation and the quantum theory of potential scattering.

INTRODUCTION

Our investigation is devoted to an inhomogeneous boundary-value problem defined by a linear partial differential equation with separable variables $x$ and $y$:

\[ (l_x + l_y)Z(x, y) = F(x, y), \]  

where $l_x$ and $l_y$ are differential expressions in the coordinates $x$ and $y$ and $F$ is the known and $Z$ is the unknown of $x$ and $y$. In (1), $Z(x, y)$ satisfies boundary conditions that are also separable with respect to $x$ and $y$. In Section 1 it is assumed that (1) is the wave equation (or a system of such equations) for a stationary field [the time factor $\exp(-loot)$ where $\omega = 2\pi f$, the oscillation frequency, is omitted] described by a function of complex amplitudes $Z(x, y)$. In this case our method can be interpreted as an expansion of the solution of (1) in waves that travel along one of the coordinates $x$ or $y$ with a more or less complicated distribution of the amplitudes along the other coordinate ($y$ or $x$, respectively), i.e., along the wave front $x = \text{const}$ or $y = \text{const}$. Each "partial" wave is characterized by a wave number and the form of the amplitude distribution along the front. Thus, for two-dimensional problems there exist two types of expansion in partial waves. For reasons that will be explained below, we shall call them expansions in normal waves and, in the general case, spectral-source-like expansions or, in abbreviated form SpySox and SpSoy expansions, where Sp (spectral) refers to the coordinate along the wave front and So (source-like) refers to the coordinate along which the wave travels.

Expansions in normal waves were already employed by Rayleigh in problems without loss of wave energy along the coordinate of the wave front. However, for present-day problems of the theory of propagation of radio waves around the Earth expansions in normal waves with an unbounded front, which take into account energy loss due to radiation, are of more importance. To construct them, Sommerfeld [1], Poincaré [2], and Watson [3] transformed the expansion in normal waves that travel along one direction into an expansion in waves along the other direction. This was done by means of an integral transformation, which is known as the Sommerfeld-Watson transformation. In the investigations of Regge it has been used for the same purpose in the theory of the potential scattering of particles.

After the appearance of [6, 7], it became clear that an adequate mathematical apparatus for these expansions is the spectral theory of differential linear operators; this makes it possible to obtain both types of expansions without the use of the Sommerfeld–Watson transformation (the method of normal waves) and also to make a qualitative investigation of the solutions obtained. For boundary-value problems with radiation losses and in a propagation "medium," one must use the theory of nonself-adjoint operators and, in the general case, operators that are also singular. Such an approach is adopted in the present paper.

1. Spectral-Source-Like Expansions in the Light of the Spectral Theory of Operators

A. We begin with the classical problem of the diffraction of radio waves around a perfectly conducting Earth in a vacuum. Introducing spherical coordinates \( r, \theta, \) and \( \phi \) with pole in the center of the Earth, we place an elementary Hertz electric dipole with moment \( F_0 \) at the point \( \theta = 0, \ r = b \geq a, \) where \( a \) is the radius of the Earth. The field of the waves excited by this dipole is described by the wave equation

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + r^2 \frac{\partial^2 U}{\partial r^2} + k^2 U = 0,
\]

where \( U \) is the Hertz potential and \( k = \omega / c, \) where \( c \) is the velocity of light.

The solution of (1') is sought for the boundary conditions

\[
\partial U / \partial r = 0 \ (r = a); \ U \sim e^{-kr} \ (r \to \infty).
\]

It must also satisfy a singularity at the point of the emitter. The original solution of this problem, which was found by MacDonald as early as 1903, consisted of an expansion of the field in partial waves traveling along the radius. We shall refer to such an expansion in an abbreviated form as the SP\( \theta \)So \( r \) expansion.

Following MacDonald, we represent the field produced by the Hertz dipole in vacuum in the form of a SP\( \theta \)So \( r \) expansion:

\[
U_0 = \sum_{n=0}^{\infty} \left[ \frac{h_0^{(1)}(kr)}{r} \right] \left[ \frac{h_n^{(1)}(kb)}{r} \right] P_n(\cos \theta).
\]

Then the total field, which MacDonald obtained by fitting the primary and secondary fields at \( r = a, \) can be expressed in the form

\[
U = \frac{F_0}{krb} \sum_{n=0}^{\infty} \left\{ \frac{h_n^{(1)}(kb) D_n(a',r)}{r} \right\} P_n(\cos \theta),
\]

where \( J_n = (h_n^{(1)} + h_n^{(2)}) / 2 \) and \( h_n^{(1,2)}(kr) = \sqrt{\pi kr} H_n^{(1,2)}(kr) \) are the modified Hankel functions; \( P_n(\cos \theta) \) are the Legendre polynomials, and \( D_n(x,y) = h_n^{(1)}(x)h_n^{(2)}(y) - h_n^{(2)}(x)h_n^{(1)}(y) \) are two-argument functions; the derivatives are indicated by primes. The forms of the partial waves in (2) are determined by Legendre polynomials and the phase factors for \( r > b \) by the function \( h_n^{(1)}(kr). \)

Even at long radio wavelengths the expansion (2) does not converge well and for almost two decades the best mathematicians at the beginning of the twentieth century sought to find a more effective solution of this problem. In this connection we must mention the paper of Poincare's [2] and especially Watson's work [3], who transformed (2) into a new series, now interpreted as an expansion in waves traveling along the angular coordinate \( \theta \) (we shall refer to it as the SP\( \theta \)So \( \theta \) expansion). For large \( \theta, \) a single wave with the least damping is predominant in this expansion. According to Watson, this transformation is done as follows. The series (2) is replaced by an integral with respect to the variable \( \nu = n + 1/2: \)

\[
U = \frac{2\pi F_0}{k \rho^2} \int_{\Gamma_\nu} \frac{\nu d \nu}{\sin \nu \cos \nu} \left\{ \frac{h_n^{(1)}(kb) D_{\nu-1/2}(a',r)}{r} \right\} \left\{ \frac{h_n^{(2)}(kr) D_{\nu-1/2}(a',b)}{r} \right\} P_{\nu-1/2}(\cos \theta),
\]

where \( \Gamma_\nu \) is a contour that surrounds the poles of the integrand, \( \nu_n = n + 1/2; \ n = 0, 1, 2, 3, \ldots \) Apart from the poles \( \nu_n \) on the real axis, there is an infinite number of poles in the complex \( \nu \)-plane defined by the equation

\[
\frac{d}{dr} h_n^{(j)}(ka) = 0, \quad j = 0, 1, 2, 3, \ldots .
\]

Therefore, deforming the contour \( \Gamma_\nu, \) we can calculate (3) by taking the residues with respect to these poles at the points \( \nu_j. \) Then