A criterion for the commutant of a quantized field to be algebraically closed

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A necessary and sufficient condition is formulated for the commutant [4-5] of a given quantized field to be algebraically closed (i.e., closed with respect to algebraic operations). If the field satisfies the usual Wightman axioms, the assumption that its commutant is a * algebra implies that this * algebra is abelian.

1. Introduction

This paper continues the author's earlier investigations [4-5]; the concepts and results of these papers will frequently be quoted without additional explanation and the introduction to the present paper will be kept correspondingly brief.

In the investigations [4-5] an attempt was made to construct the theory of representations of a topological (non-Banach) algebra with an involution (called A) by linear operators on a Hilbert space. The value of such an investigation is not merely mathematical since the formalism of the theory of representations is the natural language of Wightman axiomatics [1, 5] and the theory of representations of canonical commutation relations.

We begin with the basic definitions.

We shall say that a * representation R of a * algebra A is defined on a Hilbert space H if every \( a \in A \) is associated with a linear operator \( r(a) \) on \( H \) and the correspondence \( a \rightarrow r(a) \) possesses the following properties (henceforth, we shall omit the asterisk in front of the words "algebra" and "representation"):

1. All the operators \( r(a) \) are defined on one and the same domain \( D_R \) which is linear and dense in \( H \).
2. The domain \( D_R \) is invariant with respect to the operators \( r(a) \):
   \[ r(a)D_R \subset D_R, \forall a \in A; \]
3. The correspondence \( a \rightarrow r(a) \) establishes a homomorphism between \( A \) and the set of linear operators on \( H \):
   \[ r(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 r(a_1) + \lambda_2 r(a_2), r(a_1 a_2) = r(a_1) r(a_2) \]
   for all \( a_1, a_2 \in A \) and all complex numbers \( \lambda_1, \lambda_2 \).
4. There exists a domain \( D \subset D_R \) which is linear and dense in \( H \) and such that all the adjoint operators \( r(a)^* \) are defined on \( D \) and coincide on \( D \) with the operators \( r(a^+) \):
   \[ r(a)^*|_D = r(a^+)|_D. \]

If \( r(a)^* \supset r(a^+) \), \( \forall a \in A \), one can take \( D = D_R \); in this case, we shall say that the representation is symmetric. The adjoint of a given representation [we shall call it \( R^* \) and denote its operators by \( r^{(i)}(a) \)] is defined as follows:

\[ D_{R^*} = \bigcap_{a \in A} D_{r^{(a)}(a^*)}; \]
r(\(t\))(a) is the restriction to \(D_{R^*}\) of \(r(a^*)\) (it is shown in [4] that the set of operators defined in this manner does indeed form a representation in the sense of the above definition).

If two representations \(R_1, R_2\) are defined on a given Hilbert space \(H\) and \(D_{R_1} \supset D_{R_2},\ r_1(a) \supset r_2(a), \ V a \in A,\) we shall say that \(R_1\) is an extension of \(R_2\) (and write \(R_1 \supset R_2\)).

Let \(R^{**} \equiv (R^*)^*;\) we shall denote the operators of \(R^{**}\) by \(r^{(0)}(a)\). For a symmetric representation \(R \subset R^{**} \subset R^* [4].\) We shall say that a representation is selfadjoint if \(R = R^*\) and essentially selfadjoint if \(R^{**} = R^*\). The narrow commutant \(R_{na}'\) of the representation \(R\) on \(H\) is defined as the set of all bounded operators \(B\) on \(H\) such that

\[
BD_n \subset D_n, B'D_n \subset D'_n. \\
Br(a) \subset r(a)R, B'r(a) \subset r(a)B', \ V a \in A.
\]

The narrow commutant of any representation is an algebra but does not possess any property of being closed.

The commutant \(R'\) of a symmetric representation \(R\) is defined as the set of all bounded operators \(B\) on \(H\) such that

\[
\langle x, Br(a)y \rangle = \langle r(a)x, By \rangle, \ V a \in A; x, y \in D_n.
\]

The commutant \(R'\) of any symmetric representation is closed in the weak operator topology and contains \(R_{na}'\); however, it is not an algebra in the general case; it is closed with respect to operations of addition, multiplication by a number, and taking the adjoint, but it is not closed with respect to the operation of multiplication of two elements, i.e., \(B_1B_2 \in R'\) does not imply \(B_1B_2 \in R'\).

The following results are proved in [4]: the commutant of a selfadjoint and an essentially selfadjoint representation is a weakly closed algebra of bounded operators on \(H\) (a von Neumann algebra) and \(R' = R_{na}'\) for a selfadjoint representation and \(R' = (R^{**})_{na}'\) for an essentially selfadjoint representation.

### 2. Criterion for the Commutant to be Algebraically Closed

**Proposition 1.** The commutant \(R'\) of a symmetric representation \(R\) is an algebra if and only if there exists a symmetric extension \(\bar{R} \supset R\) such that \(R' = R' = R_{na}'\).

**Proof.** The sufficiency is obvious since the narrow commutant of any representation is an algebra. We shall prove the necessity by directly constructing the desired extension \(\bar{R}\).

Thus, suppose that \(R'\) is an algebra. We define \(D_{R'}\) as the linear hull of the set of all vectors of the form \(Bx\) for arbitrary \(B \in R', x \in DR\). Now \(D_{R'}\) is linear by construction and dense in \(H\) since \(D_{R} \supset D_{R'}\) (because \(R'\) always contains the identity operator). On the other hand, \(D_{R'} \subset D_{R'^*}\), since \(BD_{R} \subset D_{R'^*}\) for any \(B \in R' [4]\).

Let \(\bar{r}(a)\) be the restriction of \(r(\(t\))(a)\) to \(D_{R'}\). We wish to show that the set of operators \(\bar{r}(a)\) forms the desired symmetric extension of \(R\).

Let us verify that \(D_{R'}\) is invariant with respect to the operators \(\bar{r}(a)\) (property 2 of a representation). To this end, we use a relation proved in [4]:

\[
r(a)Bx = Br(a)x, \ V a \in A, B \in R', x \in D_{R'^*},\n\]

which can be rewritten for \(x \in D_{R}\) in the form \(r(a)Bx = Br(a)x, \) since \(R^{**} \supset R.\) Therefore, for all \(B_i \in R', x_i \in DR, i = 1, \ldots, n,\) we have

\[
\bar{r}(a) \sum_{i=1}^{n} B_i x_i = r(a) \sum_{i=1}^{n} B_i x_i = \sum_{i=1}^{n} B_i r(a)x_i \in D_{R'},
\]

since \(r(a)x_i \in DR\) by virtue of the invariance of \(DR\) with respect to the operators \(r(a).\) The upshot is \(\bar{r}(a)D_{R} \subset D_{R'} V a \in A,\) as we wished to prove.

Property 3 of a representation is satisfied since \(\bar{R} = \{\bar{r}(a); a \in A\}\) is a restriction of the representation \(R^*.\) It remains to show that property 4 is satisfied. To do this, it is sufficient to show that