A METHOD OF FINDING THE JOST MATRIX

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The problem of finding the Jost matrix by solving a Riemann boundary-value problem in the presence of mixing of partial waves in the S matrix is investigated. An algorithm for solving the problem is given for the case when the cotangent of the mixing angle is represented in the form of a rational function.

There exists a number of physical problems that can be reduced to a one-dimensional Riemann boundary-value problem (see, for example, [1-9]). These include the problem of finding the Jost function. In the present paper, we discuss the question of finding the Jost matrix in the presence of mixing of the partial waves. This case is completely different from the diagonal case, for which the problem can be solved completely and the answer reduced to quadratures. In the presence of mixing, the problem becomes essentially a matrix problem, and one can then reduce it only to a system of Fredholm equations [9-13]. Using the method proposed in [14, 15], we construct an explicit solution of the problem in which the cotangent of the mixing angle is specified in the form of a rational function.

It is well known (see, for example, [16]) that the Jost matrix can be found as solution of the Riemann boundary-value problem for a half-plane:

\[ F^+(k) = S^{-1}(k)F^-(k), \quad \text{Im} k = 0, \quad -\infty < k < \infty. \]  

(1)

Here, \( F^+(k) \) is analytic in the upper half-plane, \( F^-(k) \) in the lower half-plane, and \( k \) is the modulus of the momentum in the center of mass system. The scattering matrix \( S(k) \) satisfies the condition \( S(-k) = S^*(k) = S^{-1}(k) \). Having in mind the use of the results of the present paper for dispersion calculations of the electromagnetic structure of the deuteron [6-8], we shall assume in what follows the case of triplet neutron–proton scattering.

If the solution of the boundary-value problem (1) is to represent the Jost matrix, it must satisfy the additional relations \( F^-(k) = F^+*(k) \), the determinant of \( F(k) \) at the point \( k = i\infty \) of the bound state must have a zero of first order, and in asymptotia \( F(k) \) must go over into the unit matrix.

We write the S matrix in the Blatt–Biedenharn parametrization

\[ S(k) = \begin{pmatrix} \cos \varepsilon & \sin \varepsilon \\ -\sin \varepsilon & \cos \varepsilon \end{pmatrix} \begin{pmatrix} \exp(2i\delta_s) & 0 \\ 0 & \exp(2i\delta_d) \end{pmatrix} \begin{pmatrix} \cos \varepsilon & -\sin \varepsilon \\ \sin \varepsilon & \cos \varepsilon \end{pmatrix}. \]  

(2)

Here, \( \delta_s(k), \delta_d(k) \) are the S and D phase shifts, respectively, and \( \varepsilon = \varepsilon(k^2) \) is the mixing angle, with

\[ \delta_s(-k) = \delta_s(k), \quad \delta_s(\infty) = 0, \quad \delta_s(0) = \pi, \quad \delta_d(0) = 0. \]  

(3)

The total index of the boundary-value problem (1)-(3), i.e., the index of the determinant \( \det S^{-1} = \exp[-2i(\delta_s + \delta_d)] \), is \(-2\). In what follows, we shall assume that \( \cot \varepsilon(k^2) = \mathcal{P}(k^2)\mathcal{Q}^{-1}(k^2) \), where \( \mathcal{P}(k^2) \) and \( \mathcal{Q}(k^2) \) are polynomials.

We first "remove" the bound state from the S matrix, i.e., we reduce it to a quantity \( S_N \) having zero total index. For this, we use the operator \( R(x) \) introduced in [9]: \( S_N = R(k)S(k)R^{-1}(k), \quad F_N(k) = R(k)F(k) \),

where \( R(k) = E - T \frac{2i\varepsilon}{k + i\varepsilon}, \quad T = \frac{1}{1 + c^2} \begin{pmatrix} 1 & -c \\ -c & c^2 \end{pmatrix} \), \( c = \tan \varepsilon(\infty) \). Here, \( E \) is the unit matrix, and \( c \) is a constant which determines the asymptotic behavior of the ratio of the S wave to the D wave in the deuteron. Then (1) reduces to the problem

\[ F_N(k) = S_N^{-1}(k)F_N^-(k). \]  

(4)

We now transform the boundary-value problem (4) to the boundary-value problem \( F_N(k) = \Omega(k)U(k) \quad F_N^-(k) \), a method of solution for which is given in [14]; \( \Omega(z) \) (respectively, \( U(z) \)) are matrices whose elements are functions that are holomorphic in the upper (respectively, lower) half-plane except for a finite


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number of points at which they can have poles. The determinants $\det \Omega(k)$ and $\det U(k)$ can have only a finite number of zeros in their analyticity domains. Solving in the standard manner (see, for example, [16, 17]) the two auxiliary boundary-value problems $f_+(k) = \exp(-2i6a)f_+(k)$, $f_-(k) = \exp(-2i6a)f_-(k)$, whose indices are $-2$ and $0$, respectively, we reduce the boundary-value problem (4) to the form

$$\Omega^{-1}(k)F_+(k) = U(k)F_-(k), \quad U(k) = [\Omega^{-1}(k)]^*, \quad \Omega^{-1}(k) = \begin{pmatrix} f_+ & 0 \\ 0 & f_- \end{pmatrix} \begin{pmatrix} Q & -P \\ P & Q \end{pmatrix} R(k).$$

It can be seen from the construction that $\Omega^{-1}$ and $U$ do not have singularities in the upper and lower half-planes, respectively, and therefore the solution of the problem reduces to removing the zeros of the determinants of the matrices $\Omega^{-1}$ and $U$ in these half-planes. One can show that the position of the zeros is determined by the roots of the equation $p^2 + q^2 = 0$.

To construct the canonical solution, we successively multiply Eq. (6) from the left by a matrix which removes the zeros of $\det \Omega^{-1}$ and $\det U$. Suppose that at $k = a$ the determinant of the matrix $\Omega^{-1}(k)$ vanishes. We choose a matrix $Z(k)$ such that its determinant is equal to $(k - a)^{-2}$, its elements do not have other poles apart from $k = a$, and the product $Z(k)\Omega^{-1}(k)$ is holomorphic for $k = a$. Then we proceed in the same manner with the remaining zeros of the determinants of the matrices $Z(k)\Omega^{-1}(k)$ and $Z(k)U(k)$. The explicit form of this "correcting" matrix $Z$ is given in [15]. We denote by $A(k)$ the matrix that we intend to correct. Suppose the elements of $A(k)$ are holomorphic in the neighborhood of the point $k = a$. It is easy to show that

$$Z(k) = \begin{pmatrix} 1 \\ (k-a)\text{Sp} A(a) + \bar{A}(a) \end{pmatrix}$$

is the required "correcting" matrix; here, $\bar{A} = (\det A)A^{-1}$. By successive removal of the zeros of the determinants by means of matrices of the type (6), we arrive at the problem $\Omega_1F_+ = U_1F_-, U_1(k) = \Omega_1(k)$. The matrices $\Omega_1$ and $U_1$ do not have singularities in the upper and lower half-planes, respectively, and their determinants do not vanish there. The solution of the problem (4) will have the form $F_+(k) = \Omega_1^{-1}(k)F_-(k) = U_1^{-1}(k)$. Going back to the original problem (1), we obtain the final expression $F_+(k) = R^{-1}(k)\Omega_1(k), F_-(k) = F_-(k)$.

We now make two remarks concerning the partial indices of the problem. It is well known that these indices can be determined only after the canonical solution has been found. It has however been conjectured [14] that the partial indices are equal to the indices of the roots of the characteristic equation, which in our case has the form $\lambda^2 - \text{Sp} S^{-1} + \det S^{-1} = 0$, and its roots are $\lambda_i = \exp(-2i6a)$ and $\lambda_a = \exp(-2i6b)$. The partial indices in this case do not depend on the mixing angle and are determined solely by the behavior of the phase shifts. Such a result is physically justified since the difference between the phase shifts at the origin and at infinity, which determines our index, can be related by means of Levinson's theorem to the number of bound states in the given channel. It would be of interest to obtain this result rigorously.

A second remark concerns the investigation of the stability of the partial indices. It is known [10] that the partial indices of the coefficient are stable in the class of arbitrary perturbations if and only if $\kappa_1 - \kappa_2 \leq 1$. Unfortunately, this general result is not completely useful from the point of view of the physical problem of finding the Jost matrix, since the physical formulation of the problem strongly restricts the class of possible perturbations, and general stability criteria are of no use for this problem. The investigation of the stability of the partial indices in the presence of rigorously satisfied restrictions, for example, unitarity, as in our case, would also be very interesting.

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**LITERATURE CITED**