SPECTRAL PROPERTIES OF INVISCID SOLUTIONS OF THE BURGERS EQUATION WITH RANDOM INITIAL CONDITIONS

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The Burgers equation with random self-similar initial conditions is investigated numerically in the inviscid limit by a parallel fast Legendre transform algorithm, using Connection Machine CM-200. The use of this equation for solving the problem of nervous impulse propagation through axons is discussed. An attempt is made to simulate recent experiments where the form of the density of propagated nerve impulses, which initially had a power spectrum close to a white noise distribution, appeared similar to the triangular pulses that arise in the inviscid Burgers equation and where the $1/f$ power law was observed on scales larger than the typical time interval between pulses. It is shown that in the inviscid Burgers equation model the power spectra for different types of initial conditions in the developed "Burgers turbulence" regime (i.e., at a sufficiently large time) consists of two parts with a rather sharp transition between them: The spectrum virtually coincides with the initial spectra for low wavenumbers, and the $1/f^2$ law holds for high wavenumbers. There is no interval with an intermediate power law dependence such as $1/f$. It is inferred that the true $1/f$ spectrum of nerve impulses propagating through axons cannot be explained in terms of the Burgers equation model and that other mechanisms must be taken into account.

It is well known that the Burgers equation with random self-similar initial conditions has many applications in a variety of fields from acoustic waves to traffic currents and the Universe's large-scale structure formation [1,2]. One interesting application is the propagation of a nerve impulse through axons [3]. In [3], the authors investigated experimentally some properties of the spectra of impulses with a temporal distribution that is initially close to white noise in a wide frequency range. They found that the form of the impulse density is similar to the triangular pulses that arise in the Burgers equation and that the $1/f$ power law occurs on scales larger than the typical time interval between impulses. Using the conclusions drawn in [3], we attempt at simulating this process numerically in terms of the 1D Burgers equation

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \mu \frac{\partial^2 v}{\partial x^2}; \quad v(x, t = 0) = v_0(x)$$

where $\mu$ is the viscosity coefficient. The solutions of the Burgers equation with random initial conditions display two mechanisms, which are also inherent in the real turbulence: the nonlinear transfer of energy over the spectrum and the viscous damping in the small scale region. Equation (1) was proposed by Burgers [1] as a one-dimensional model of fluid turbulence. More recently, it was shown that this equation is useful for description of a wide variety of nonequilibrium phenomena when parity invariance holds [4]. The Burgers equation also applies to nonlinear acoustics, nonlinear waves in thermoelastic media, the dynamics of growing interfaces, modeling the velocity and mass distribution for large-scale structures in the Universe, and in many other systems where dispersion is negligibly small compared with nonlinearity (see [2] and references therein).

This model thus includes the nonlinear action and dissipation in the nerve impulse density propagation. We use random initial conditions for nerve impulse density (velocity in terms of the usual Burgers equation)
of the following types (we write the expressions in the discrete form in which they are used to calculate functions determined on a uniform grid \( x_i = iL/N \) of \( N \) points in the range \( 0 \leq x \leq L \))

**type 1, white noise:**
\[
v_0^w(z) = -\frac{\partial S_0(x)}{\partial z}; \quad S_0^w(x_j) = \sum_{i=0}^{j} g_i
\]

**type 2, Brownian motion (integral of white noise):**
\[
v_0^b(x_j) = \sum_{i=0}^{j} g_i
\]

**type 3, fractional white noise (the integral of which is fractional Brownian motion):**
\[
v_0^f(x) = \sum_{k} u_k e^{ikz}, \quad k = \frac{2\pi n}{L} (n = 0, \pm 1, \pm 2, \ldots).
\]

Here, \( S_0(z) \) is the potential (action) for the initial velocity, \( g_i \) are independent random numbers with the Gaussian probability distribution function, and \( u_k \)'s are complex Gaussian random variables with variance \( \langle |u_k|^2 \rangle \propto |k|^{-1-2h} \), where \(-1 \leq h \leq 0\) and \( u_k \) are assumed to be chosen independently, except that they must satisfy the Hermitian symmetry \( u_{-k} = u_k^* \). Under random self-similar initial conditions, we can always consider sufficiently large spatial and temporal scales and then rescale the variables in a fashion such that the rescaled viscosity become arbitrary small.

In the limit of zero viscosity, the solution of (1) for the velocity field is given by
\[
v(z, t) = \frac{x - y(z, t)}{t}; \quad v(z, t) = -\frac{\partial S(z, t)}{\partial z}
\]
where \( y(z, t) \) is the coordinate of the absolute maximum of the function
\[
G(x, y, t) = S_0(y) - \frac{(x - y)^2}{2t}; \quad S_0(x) = \int_{y}^{x} v_0(y) dy
\]

We have therefore solved the inviscid Burgers equation, using a discrete grid with up to \( 2^{30} \) points. In the numerical experiments, we used a parallel version of the fast Legendre transform (FLT) algorithm implemented on the parallel supercomputer Connection Machine CM-200 at the Center for Parallel Computers in Stockholm [5,6,8]. This algorithm uses the property that the inverse Lagrangian function \( y(z, t) \) is a nondecreasing function of \( z \) and the specific low-level instructions of the Connection Machine, which allows simultaneous partial maximization operations over specified divisions of the range, which are known exactly only at the run time. The most primitive computation of the maximum of (3) requires about \( O(N^2) \) operations on a sequential computer for \( N \) points. The serial FLT requires about \( O(N \log N) \) operations in 1D problems and about \( O(N^2 \log^2 N) \) operations in 2D problems when an \( N \times N \) grid is used [7]). Our parallel FLT performs \( O(\log^2 N) \) operations on an ideal parallel computer with an unlimited number of physical processors connected in a hypercube network (this type of connection is used in CM-2 and CM-200 models). The Connection Machine simulates an arbitrary number of processors as their virtual finite number (on our machine 8192). Our algorithm will therefore eventually come, linearly with the ratio of initialized gridpoints \( (N) \), to the actual number of physical processors. For the class of one-dimensional problems in question, the computational time including input, output, and Connection Machine initialization is much less than 1 min for a total number of gridpoints up to \( N = 2^{30} \).

It is well known [1,2] that solutions of the Burgers equation with arbitrary initial conditions for a sufficiently large time represent a sequence of triangular pulses, which determines the asymptotic behavior of its power spectra in the high-frequency limit as \( 1/f^2 \). For the case of random self-similar initial conditions, it was shown both analytically and numerically [2,5,6] that the asymptotic behavior of these spectra and, moreover, the solutions of the Burgers equation themselves are critically dependent on the large-scale