THE EXACT VALUE OF RELAXATION TIME FOR A DYNAMIC SYSTEM WITH NOISE DESCRIBED BY AN ARBITRARY SYMMETRIC POTENTIAL PROFILE

A. N. Malakhov and A. L. Pankratov

We show that the well-known Pontryagin's formula for the average time of first attainment by a Brownian particle of the absorbing boundary can also be used to obtain the exact value of the average relaxation time for the nonequilibrium state of a nonlinear dynamic system with noise and an arbitrary symmetric potential profile.

1. Estimating the times required by the nonequilibrium state of a nonlinear dynamic system with noise to reach equilibrium is a timely problem for a wide class of physical, chemical, and biological problems such as, for example, the switching velocities of bistable systems including Josephson's logic elements, stochastic resonance, kinetics of chemical reactions, etc. (see, for example, [1-4]).

In essence, this problem is reduced to finding the evolution velocity for the probability density $W(x, t)$ of a Brownian particle traveling in a force field with potential $\Phi(x)$ in the regime of high friction, which is described by the Fokker-Planck equation (FPE) (see, for example, [1-4])

$$\frac{\partial W(x, t)}{\partial t} = -\frac{\partial G(x, t)}{\partial x} + \frac{1}{B} \left\{ \frac{\partial}{\partial x} \left[ \frac{d \varphi(x)}{dx} W(x, t) \right] + \frac{\partial^2 W(x, t)}{\partial x^2} \right\}$$

(1)

with the initial and boundary conditions:

$$W(x, 0) = \delta(x - x_0) \quad \text{and} \quad W(\pm\infty, t) = 0.$$  

Here $B = h/kT$, $G(x, t)$ is the probability current, $h$ is the viscosity, $k$ is Boltzmann's constant, $T$ is the temperature, and $\varphi(x) = \Phi(x)/kT$ is the prescribed dimensionless potential profile such that $\varphi(\pm\infty) = \infty$.

To find the evolution velocity $W(x, t)$, we must know the nonstationary solution of the FPE, which is the main mathematical difficulty in the problem studied. In the literature, to solve the FPE, the method of expansion of the Fokker-Planck operator in terms of the eigenfunctions and the search for a minimum nonzero eigenvalue which is taken as the evolution velocity [4, 5] are used most frequently.

At the same time, there exists the well-known Pontryagin's formula [6] for the average time of first attainment (ATFA) by the Brownian particle of the absorbing boundary located at the point $x = L > x_0$

$$T(x_0, L) = B \int_{x_0}^{L} e^{\varphi(x)} \int_{-\infty}^{\varphi(v)} e^{-v} dv dx,$$

(2)

which presents the ATFA directly through the given function $\varphi(x)$ of the potential profile.

2. Note that Eq. (2), along with the ATFA, also provides the value of the average relaxation time for a dynamic system with noise, which has an arbitrary potential profile symmetric with respect to the point $L$. 

From Eq. (1), for the Laplace transform of the probability density \( Y(z, s) = \int_0^\infty W(z, t)e^{-st}dt \), one easily obtains the following equation:

\[
\frac{d^2Y(z, s)}{dz^2} + \frac{d}{dz} \left[ \frac{d\varphi(z)}{dz} Y(z, s) \right] - sBY(z, s) = -B\delta(z - z_0).
\]

Let us match the symmetry point of the dimensionless potential profile with the origin so that \( \varphi(-z) = \varphi(z) \) and assume that \( z_0 < 0 \). It is noteworthy that the probability current in the Laplace transform has the form

\[
\dot{G}(z, s) = \int_0^\infty G(z, t)e^{-st}dt = -\frac{1}{B} \left[ \frac{d\varphi(z)}{dz} Y(z, s) + \frac{dY(z, s)}{dz} \right].
\]

Assume that we know two linearly independent solutions \( U(z) = U(z, s) \) and \( V(z) = V(z, s) \) of the homogeneous equation (i.e., for \( B = 0 \)) that corresponds to Eq. (3), such that \( U(z) \to 0 \) for \( z \to +\infty \) and \( V(z) \to 0 \) for \( z \to -\infty \). On the strength of the symmetry of the function \( \varphi(z) \), these independent solutions can also be chosen symmetric, such that \( U(-z) = V(z) \), \( U(0) = V(0) \), and \( [dU(z)/dx]_{x=0} = -[dV(z)/dx]_{x=0} < 0 \).

In this case, the general solution of Eq. (3) is represented as

\[
Y(z, s) = \begin{cases} 
Y_1(z) + y^-(z), & z \leq z_0 \\
Y_1(z) + y^+(z), & z_0 \leq z \leq 0, \\
Y_2(z), & z \geq 0
\end{cases}
\]

where

\[
Y_1(z) = C_1V(z), \quad Y_2(z) = C_2U(z),
\]

\[
y^-(z) = \frac{B}{W[z_0]}U(z_0)V(z), \quad y^+(z) = \frac{B}{W[z_0]}V(z_0)U(z).
\]

Here \( W[z] = U(z)(dV(z)/dz) - V(z)(dU(z)/dz) \) is the Wronskian and \( C_1 \) and \( C_2 \) are arbitrary constants which can be found from the condition of continuity of the probability density and the probability current at the origin

\[
Y_1(0) + y^+(0) = Y_2(0), \quad \dot{G}(z = -0, s) = \dot{G}(z = +0, s).
\]

Calculating the values for the arbitrary constants from this equation and substituting them into Eq. (5), we obtain the following value for the probability current in the Laplace transform \( \dot{G}(z, s) \) at the symmetry point \( z = 0 \)

\[
\dot{G}(0, s) = \frac{V(z_0)}{W[z_0]} \left[ \frac{dV(z)}{dz} \right]_{z=0}.
\]

According to [7, 8], the characteristic time of evolution for the probability density \( W(z, t) \) in an arbitrary potential profile is

\[
\Theta = \lim_{s \to 0} \frac{s\dot{\hat{P}}(s) - P(\infty)}{s[P(0) - P(\infty)]} = \lim_{s \to 0} \frac{[P(0) - P(\infty)] - \dot{G}(L, s)}{s[P(0) - P(\infty)]},
\]

where