COORDINATE ASYMPTOTIC BEHAVIOR OF THREE-PARTICLE WAVE FUNCTIONS

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A study is made of the coordinate asymptotic behavior of the wave function for a system of three nonrelativistic particles, allowance being made for (3 → 3) processes. Attention is devoted primarily to twofold scattering effects. It is shown that in the parts of the configuration space where the formally constructed scattering amplitude becomes infinite the asymptotic behavior of the (3 → 3) wave function can be described by means of a Fresnel integral.

1. Introduction

1. The asymptotic behavior of the wave function for a system of two-particles in the coordinate space can be represented by a sum of a plane and a spherical wave [1] (we assume that the motion of the unperturbed system is described by a plane wave). In the recent investigations [2, 3] a study is made of the asymptotic behavior of wave functions for processes in which an initial two-particle state can give rise to three particles (here we are concerned with functions that are usually denoted by \( \psi^{(*)} \)). Nuttal [2] shows that the behavior in this case is basically similar to the asymptotic behavior of the two-particle wave functions. To the standard terms [1] there is added only a spherical wave (corresponding to the (2 → 3) reaction) in the configuration space \( \mathbb{R}^6 \), the amplitudes of the spherical waves in \( \mathbb{R}^3 \) and \( \mathbb{R}^6 \) being described by the elements of the S matrix and the two-particle eigenfunctions. The general features of the asymptotic behavior of the wave functions for the processes (2 → n), n ≠ 3, is also known [1].

In the present paper we shall consider the wave function \( \psi \) for a system of three nonrelativistic particles with a pair interaction, taking into account (3 → 3) processes with three free particles before and after the reaction. The center of mass motion is assumed to be separated off. We obtain the asymptotic behavior of \( \psi \) in the configuration space \( \mathbb{R}^6 \). This behavior is shown to possess a number of features that have no counterparts in two-particle scattering. They are manifested in the S matrix as singularities due to single and twofold collision processes.

After the completion of the present investigation, I became acquainted with Gerjuoy's reprint [4]; he also discusses (3 → 3) scattering in the configuration space, devoting his main attention to twofold scattering effects (as in the present paper). Gerjuoy concludes incorrectly that the principal terms of the asymptotic behavior corresponding to twofold scattering have an order at infinity that does not depend on the direction. The present paper is based on a rigorous investigation of Faddeev's equations and it shows, in particular, that the order of these terms depends essentially on the direction.

In this connection we should mention McGuire's paper [5] devoted to a model of three one-dimensional particles; he notes that there is an analogy between the problem of finding the asymptotic behavior of the (3 → 3) wave function and the problem of Fraunhofer diffraction by a wedge. However, McGuire does not in fact develop this analogy with optics further in [5].

The analytic apparatus employed in the present paper to describe twofold scattering effects has much in common with the technique employed in the Fraunhofer diffraction problem.

*The potential of the interaction between the particles is assumed to decrease sufficiently rapidly.

2. The main results of the present paper are as follows. Suppose the particles are well separated before and after the interaction. Then the asymptotic behavior of $\psi$ in the configuration space $\mathbb{R}^4$ is given by the sum

$$\psi = \psi_0 + \sum_n U_n + \sum_{\alpha} U_\alpha + \mathcal{G},$$  \hspace{1cm} (1.1)$$

where the individual terms have the following meaning.

A) $\psi_0$ is a plane wave in $\mathbb{R}^4$, describes the free motion of three-particles.

B) $U_\alpha \ (\alpha = 1, 2, 3)$ is associated with the process of single collision

of the particle pair $\alpha$ and can be expressed in terms of the $S$ matrix for the two-body system.

C) $\tilde{U}_{\alpha\beta} \ (\alpha, \beta = 1, 2, 3, \alpha \neq \beta)$ is due to the singular contribution of successive individual two-particle collisions

and can be expressed in terms of the two-particle $S$ matrix. As regards its form, this term cannot be reduced to a wave of a definite type, in contrast to two-particle scattering. In the asymptotic behavior of $\tilde{U}_{\alpha\beta}$ a special role is played by the shadow boundaries, the scattering directions $\tilde{X}, \tilde{X} = X|X|^{-1}, X \in \mathbb{R}^4$, in the configuration space $\mathbb{R}^4$ which are allowed for classical particles by the momentum and energy conservation laws in the successive two-particle reactions. On the passage through these boundaries the form of $\tilde{U}_{\alpha\beta}$ changes abruptly and can be described by means of a Fresnel integral. The order of $\tilde{U}_{\alpha\beta}$ is $|X|^{-5/2}$ in the shadow, increasing to $O(|X|^{-3})$ in the illuminated region. The formally constructed three-particle scattering amplitude becomes infinite at these boundaries.

D) $\tilde{U}$ is the smooth part of the three-particle scattering, is a spherical wave in $\mathbb{R}^4$. The amplitude of this wave is determined by $S_{\alpha\alpha}$ matrix elements of the three-particle $S$ matrix and by the two-particle $S$ matrices. The subscript $0$ of the matrix element $S_{0\alpha}$ indicates three free particles.

Suppose $\psi$ is considered in a part of $\mathbb{R}^4$ in which the particles of pair $\alpha$ are not well separated, i.e., the distance between them $|x_\alpha|, x_\alpha \in \mathbb{R}^4$ after the interaction is essentially bounded. In addition to terms of the type (1.1), which can be interpreted in the same way as above, the asymptotic behavior of $\psi_\alpha$ associated with the reaction of the formation of the bound particle pair $\alpha$. To determine $\psi_\alpha$ one must specify the matrix element $S_{\alpha\alpha}$ of the three-particle $S$ matrix and the eigenfunctions of the two-particle energy operator. The subscript $\alpha$ in $S_{\alpha\alpha}$ corresponds to the bound pair $\alpha$.

We shall not give a detailed description of the analogs of the terms (1.1) for bounded $|x_\alpha|$. In specific problems the technique employed in this paper enables one to find the form of these terms in any part of the configuration space with the necessary accuracy.

We shall leave out of this account entirely processes in which the particles are not well separated before the interaction. However, one can also separate typical terms in this case using the methods employed in the present paper.

3. The paper is divided up as follows. In the next section we introduce the main notation and describe $\psi$ in the momentum space, following basically Faddeev [6].

In the third section we give explicit expressions for $\psi_0$ and $U_\alpha$. We construct the asymptotic behavior of the smooth part $\bar{U}$ [see (3.13)] and the terms $\psi_\alpha$ [see (3.6)]. A number of preparatory formulas for $\tilde{U}_{\alpha\beta}$ are derived.

In 4 we derive Eqs. (4.10)-(4.12), which describe the asymptotic behavior of $\tilde{U}_{\alpha\beta}$. At the end of this section we consider the case of scattering "along a straight line" (the particle momenta lie on a straight line