TIME VARIATIONS OF CERTAIN CHARACTERISTICS
OF PULSES PROPAGATING IN AN UNBOUNDED
DISPERSE MEDIUM

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An equation system is derived from the radiation transfer equation for the space moments of pulses scattered by fluctuations in a statistically homogeneous, stationary medium. The lower space moments describe the volume-averaged characteristics of a pulse, such as the energy spectrum, radius vector of the center of gravity, and the mean-square broadening of the pulse. The equations for the moments are solved for the case of radiation scattering by small-scale fluctuations of the electron concentration in a cold plasma. Some estimates are given for the frequency spectrum of a pulse in the case of scattering by large-scale fluctuations.

1. INTRODUCTION

We shall consider the scattering of pulses of high-frequency radiation by electron-density fluctuations in a statistically homogeneous, stationary medium. The radiation is described by a scalar transfer equation [1, 2]:

\[
\left[ \frac{1}{v(\omega, n)} \frac{\partial}{\partial t} + n \nabla + \alpha(\omega, n) \right] I(r, t, \omega, n) = \sigma(\omega', n' \rightarrow \omega, n) I(r, t, \omega', n') + q(r, t, \omega, n),
\]

where \( I(r, t, \omega, n) \) is the ray intensity; \( v(\omega, n) \) is the absolute value of the group velocity for the given waves; \( n = v/n \) is the unit vector having the direction \( v(\omega, n) \); \( \sigma(\omega', n' \rightarrow \omega, n) \) is the cross section for scattering from the direction \( n' \) and frequency \( \omega' \) to the direction \( n \) and frequency \( \omega \); \( \alpha(\omega, n) \) is the attenuation factor. The function \( q(r, t, \omega, n) \) describes the radiation sources. Except where noted in comments on the equations, we take integration to be carried out with respect to the primed variables throughout their entire domain of variation everywhere; integration with respect to \( n' \) is identical to integration with respect to the solid angle \( d\Omega_n \).

For scattering by time-independent fluctuations, \( \sigma(\omega_1, n_1 \rightarrow \omega, n) \sim \delta(\omega - \omega_1) \). Here the various frequency components of the ray intensity \( I(r, t, \omega, n) \) may be treated independently. In the general case, Eq. (1) formally interrelated the values of \( I(r, t, \omega, n) \) at all frequencies, but, as is clear from physical considerations, this equation is only applicable in a fairly high-frequency region of the spectrum for \( \omega \gg \tau^{-1} \sim \nu \alpha \), where \( \tau \) is the characteristic scale of variation of the ray intensity with time.

For scattering of a naturally polarized transverse mode in a statistically isotropic cold plasma, the scattering cross section has the following form for the elementary model of the media [3]:

\[
\sigma(\omega_1, n_1 \rightarrow \omega, n) = r_0^2 \frac{1}{2} \frac{(n_1 \omega_1^2 / \omega)^2}{(n_1 \omega_1^2 / \omega)} \sqrt{\frac{\varepsilon(\omega)}{\varepsilon(\omega_1)}} \frac{\langle \delta N^2 \rangle (k - k_1)}{2\pi} \left|_{k = k_0, k_1} \right.
\]

Here \( r_0 = e^2/(mc^2) \) is the classical electron radius; \( \varepsilon(\omega) = 1 - (\omega_\text{p}^2 / \omega^2) \); \( \omega_\text{p} = \sqrt{4\pi Ne^2/m} \) is the plasma frequency; \( k_\omega = (\omega/c)\sqrt{\varepsilon(\omega)} \); \( k = (\omega, k) \); \( \langle \delta N^2 \rangle (k) \) is the space-time spectrum of electron-concentration fluctuations:

\[
\langle \delta N^2 \rangle (k) = \int \delta N(r, t) \delta N(r + \rho', t + \tau') \exp \left( -i\rho' + i\tau' \right) d\rho' d\tau'.
\]

Below we shall only consider high-frequency waves for which \( \omega, \omega' \gg \omega_\text{p} \). For such waves we may let \( \varepsilon(\omega) \approx 1, \nu(\omega) = c \sqrt{\varepsilon(\omega)} \approx c \) in (1) and (2). Moreover, we shall neglect the frequency variation in the attenuation factor \( \alpha \) for the given waveband, taking \( \alpha \approx \) const.

Equations of the form (1) have recently received vigorous study, principally in connection with the theory of neutron transport [4, 5]. Even for an unbounded medium where there is no need to specify boundary conditions in order to solve (1) (neglecting the trivial condition requiring the field to vanish at infinity), an analytic solution can be obtained in general form only for a special (factorized) form of cross section: 

\[ \mathcal{R}(\omega, \mu, n) = \sum_{i=1}^{n} a_i(\omega, n) b_i(\omega, n). \]  

(4)

We ordinarily employ (4) as an approximation suited to determination of quantitative estimates. As an example, the "total frequency redistribution" approximation used in astrophysics [6] corresponds to approximate representation of the cross section in the form (4) with a single term \( m = 1 \).

For scattering by electron-concentration fluctuations the scattering cross section (2) does not reduce to (4), while the approximate representation of (2) in the form (4) requires that we use a large number \( m \) of terms. Here the analytic solution is too complicated to yield estimates, and practical application of this solution requires the use of numerical methods.

Here we shall use the method of moments to investigate the characteristics of pulses propagating in an unbounded dispersive medium; this represents a trivial generalization of the method described in [5]. In place of the intensity \( I(r, t, \omega, n) \) we shall evaluate cruder integral characteristics - the space moments of the field:

\[ \mathcal{R}^m_{\text{mom}} = \int r^m I(r, t, \omega, n) \, dr, \]

(5)

where the matrix \( \mathcal{R}^m \) is defined as

\[ \mathcal{R}^m = \left[ r_1 \otimes r_2 \ldots \otimes r_m \right] = \frac{r_1 \otimes r_2 \ldots \otimes r_m}{m}, \]

(6)

and \( r_1 \otimes r_2 \) indicates the tensor product of \( r_1 \) and \( r_2 \). We assume that the pulses decay rapidly enough in space so that all moments exist.

The determination of \( \mathcal{R}^m_{\text{mom}} \) for all \( m = 0, 1, 2, \ldots \) in principle permits us to reproduce \( I(r, t, \omega, n) \), but such a procedure for obtaining \( I(r, t, \omega, n) \) is cumbersome and inconvenient. Thus, it only makes sense to employ the method of moments when the conditions of the problem are such that we need only know several of the lower moments, or when we wish to check results obtained by other methods. The first case may be encountered fairly often, since the lower moments have a simple physical interpretation and give a fairly complete description of the changes produced in a pulse owing to scattering by irregularities.

Let us take note of the physical interpretation of the simplest moments. The zeroth moment \( \mathcal{R}^0_{\text{mom}} \) has the meaning of the total energy of the components of a pulse having the frequency \( \omega \), the components propagating in the \( n \) direction at time \( t \). The first and second moments permit us to estimate, in obvious fashion, the motion of the pulse center of gravity and the spatial broadening of the pulse with time. When the spatial form of the pulse is unimportant (if the recording instrument reproduces the pulse "as a whole" - for example, integrating the field over space), the lower moments are the fundamental characteristics of the pulse.

Here we derive recursion relationships permitting us to calculate moments of any order for small-scale fluctuations; we also give certain estimates for the spectral density \( I(t, \omega) = \mathcal{R}^t_{\text{mom}} \), in the case of large-scale fluctuations. In contrast to the approximation (4), which reduces the transfer equation to an integrodifferential equation with a degenerate (factorized) kernel, the analysis described is based on approximation of the cross section by an equation having a difference (in terms of frequency) kernel.

### 2. Scattering by Small-Scale Fluctuations

Let us multiply both sides of (1) by \( \mathcal{R}^m \) and integrate the results with respect to \( dr \), assuming that there is no radiation at infinity:

\[ \left( \frac{1}{c} \frac{\partial}{\partial t} + \mathcal{R} \right) \mathcal{R}^m_{\text{mom}} = \left( \delta^i_j \mu^i - \omega, n \right) \mathcal{R}^m_{\mu^{-1}} + \mathcal{S} \mathcal{R}^m_{\mu^{-1}} + \mathcal{S}^m \mathcal{R}^m_{\mu^{-1}} - 1 \]

(7)

Here \( \mathcal{S} \) is the operator for symmetrization with respect to tensor indices; it is so defined that

\[ \mathcal{S} \mathcal{R}^m \otimes \mathcal{R} \otimes \ldots \equiv \mathcal{R}^m \otimes \mathcal{R} \otimes \ldots + \mathcal{R} \otimes \mathcal{R} \otimes \ldots + \ldots \equiv (uv) \mathcal{R} \otimes \mathcal{R} \otimes \ldots. \]  

(8)