typical for radio waves reflected from the ionosphere in vertical sounding ($\beta^2 = 3$, $R = 0.75$, $\tau = 3$ sec, $\Delta R = 0.05$, and $\Delta \tau = 2$ sec), the error $\delta(\Delta \omega)$ is less than 10% and $\delta(\beta^2)$ is less than 20%. For the model (1), it has previously been possible to determine only one parameter, namely, $\beta^2$ [1, 2], by analyzing the distribution law or the characteristic function of the process. It was shown above that a number of important parameters (9), (13), and (14) of the periodically nonstationary process (1) can be determined from the analysis of the autocorrelation function of its phase-quadrature component.

LITERATURE CITED


MOMENT AND CUMULANT FUNCTIONS OF STOCHASTIC
LINEAR SYSTEMS

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Closed equations in the diffusion approximation are derived for the moment and cumulant functions of the output signal of a general stochastic system with Gaussian fluctuations of the parameters. We consider the solution of these equations for the second- and fourth-order cumulant functions. As an example we obtain the first four cumulants of the harmonic oscillator with frequency fluctuations, as well as the cumulants of a stochastic system described by a first-order differential equation.

§1. The analysis of stochastic linear systems (dynamical systems whose parameters are random processes or fields) is very important at the present time; this refers mainly to the systems with strong fluctuations of parameters. In this field there are a large number of publications (see, for example, [1-8] and the references there). There are a number of techniques for obtaining the closed equations for the mean value and for the "output" correlation function which use the rapid fluctuation rate of the parameters and which in the end reduce to the Dyson or Bethe-Salpeter equations in one or another approximation [2-6]. Although these methods have been formally generalized for higher moments, in practice it has been possible to obtain moments higher than second only for systems described by a first-order differential equation.

Most information about the behavior of the stochastic system can be obtained from the "output" probability distribution. Unfortunately, this is usually impossible to find. At the same time, finding the first few cumulant functions is of great interest and can be sometimes accomplished for very general dynamical systems. Below we shall obtain and analyze the equations obtained in the diffusion approximation for the steady-state values of the "output" cumulant and moment functions of the $n$-th order of linear stochastic systems whose parameter fluctuations are Gaussian processes with sufficiently short correlation time. A similar problem for the correlation function was solved in [8].

§2. Let us consider a dynamical system described by an $N$-th order differential operator with fluctuating coefficients:

$$L \left( \frac{d}{dt} \right) y(t) = x(t), \tag{1}$$

where

$$L \left( \frac{d}{dt} \right) = \frac{d^N}{dt^N} + \sum_{k=0}^{N-1} a_k(t) \frac{d^k}{dt^k},$$

$$a_k(t) = \langle a_k \rangle + s_k(t), \quad \langle s_k(t) \rangle = 0.$$

The quantities $\{a_k(t)\}$ are stationary Gaussian fluctuations, $x(t)$ and $y(t)$ are the input and output signals, respectively, and we shall assume that $x(t)$ is known.

We write the output signal in the form $y(t) = \bar{y}(t) + \tilde{y}(t)$, where

$$\bar{y}(t) = \int_{-\infty}^{t} g_0(t - \tau) x(\tau) d\tau, \quad \tilde{y}(t) = -\int_{-\infty}^{t} g_0(t - \tau) \alpha_k(\tau) y^{(k)}(\tau) d\tau,$$

and the quantity $g_0(t - \tau)$ is the Green function of the unperturbed system, and obtain the following integral representation of the moment functions of the process $\tilde{y}(t)$:

$$\langle \tilde{y}(t_1) \tilde{y}(t_2) ... \tilde{y}(t_m) \rangle = (-1)^m \int_{-\infty}^{t_1} g_0(t_1 - \tau_1) d\tau_1 ... \int_{-\infty}^{t_m} g_0(t_m - \tau_m) d\tau_m \langle \alpha_k(\tau_1) y^{(k)}(\tau_1) \alpha_k(\tau_2) y^{(k)}(\tau_2) ... \alpha_k(\tau_m) y^{(k)}(\tau_m) \rangle.$$

The corresponding expressions for the cumulant functions are

$$\langle y(t_1), y(t_2), ..., y(t_m) \rangle = (-1)^m \int_{-\infty}^{t_1} g_0(t_1 - \tau_1) d\tau_1 ... \int_{-\infty}^{t_m} g_0(t_m - \tau_m) d\tau_m \langle \alpha_k(\tau_1) y^{(k)}(\tau_1) \alpha_k(\tau_2) y^{(k)}(\tau_2) ... \alpha_k(\tau_m) y^{(k)}(\tau_m) \rangle.$$

We denote for brevity $y_\alpha = y(\tau_\alpha)$ and $\alpha_{k\alpha}^\beta = \alpha_{k\alpha}(\tau_\beta)$. The brackets $\langle 1, 2, ..., m \rangle$ will denote the cumulant functions of an arbitrary set of statistically connected variables following [9, 10].

The mixed moment functions on the right-hand side of (3) are transformed by using the well-known Furutsu - Novikov expression [11, 12]:

$$\langle \alpha_k y^{(k)} \rangle = \int \mathcal{B}_{k1}(\tau_1 - \theta_1) \mathcal{B}_{k2}(\tau_2 - \theta_2) ... d\theta_1 d\theta_2,$$

$$\langle \alpha_k y^{(k)} \alpha_k y^{(k)} \rangle = \mathcal{B}_{k12}(\tau_1 - \tau_2) \mathcal{B}_{k21}(\tau_2 - \tau_1) + \int \mathcal{B}_{k1}(\tau_1 - \theta_1) \mathcal{B}_{k2}(\tau_2 - \theta_2) \mathcal{B}_{k3}(\theta_3) ... d\theta_1 d\theta_2 d\theta_3,$$

where $\mathcal{B}_{kj}(\tau) = \langle \alpha_k(t) \alpha_j(t - \tau) \rangle$ is the joint covariance function of the parameter fluctuations. It is not difficult to show that the expression for the $m$-th moment function of the output signal contains mean values of the variational derivatives of the type

$$\frac{\partial^s}{\partial \theta_1^{s_1} ... \partial \theta_m^{s_m}} \left( y^{(k_1)} ... y^{(k_m)} \right)$$

and the parity of $s$ is determined by the parity of the moment function. We note that the expansion of an even moment function $\langle \alpha_{k_1} y^{(k_1)} ... \alpha_{k_m} y^{(k_m)} \rangle$ contains unpaired terms, while an odd moment function does not.

If the correlation times of the parameter fluctuations are small in comparison with the characteristic times of the system, the correlation functions in the integrals in (5) can be replaced by the $\delta$-functions:

$$\mathcal{B}_{kj}(\tau) = \delta_{kj} \delta(\tau), \quad \mathcal{D}_{kj}(\tau) = 2 \int \mathcal{B}_{kj}(\tau) d\tau.$$

This leads to the following expressions for uncoupling in the diffusion approximation:

$$\langle \alpha_k y^{(k)} ... \alpha_{k_{2n-1}} y^{(k_{2n-1})} \rangle = 0,$$

$$\langle \alpha_k y^{(k)} ... \alpha_{k_{2n}} y^{(k_{2n})} \rangle = (2n - 1)! [ \mathcal{B}_{k1}(\tau_1 - \tau_2) ... \mathcal{B}_{k_{2n}}(\tau_{2n-1} - \tau_{2n}) ] \langle y^{(k_1)} ... y^{(k_{2n})} \rangle = \langle \alpha_k ... \alpha_{k_{2n}} \rangle \langle y^{(k_1)} ... y^{(k_{2n})} \rangle.$$

*Here and below, repeated indices are to be summed over from 0 to $N - 1$. The lower limit of integration is $-\infty$; i.e., we assume that the transient process due to the switching on of the input signal is completed.

†Here and below, the brackets $\langle ... \rangle$ denote the symmetrization over number indices and the coefficient in front gives the number of terms in the corresponding sum.