ALLOWANCE FOR SOME CORRECTIONS TO THE PARABOLIC QUASI-OPTICAL APPROXIMATION IN THE STATISTICAL DESCRIPTION OF WAVES PROPAGATING IN RANDOMLY INHOMOGENEOUS MEDIA

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1. Let the propagation of a wave in a randomly inhomogeneous layer be described by the Helmholtz stochastic equation

\[ \Delta E + k^2 E = k^2 \epsilon(x, p) E. \]  \hspace{1cm} (1)

Here \( x \) is the longitudinal and \( p \) are the transverse coordinates and \( \epsilon(x, p) \) are the random inhomogeneities of the medium, equal to zero outside the layer \( x \in [0, L] \). We will assume, in addition, that a wave equal to \( E_0(p) \) in the plane \( x = 0 \) falls on the layer on the left side.

If the inhomogeneities of the medium are large-scale while the wave incident on the layer is quasi-plane, one usually changes from the exact, stochastic, Helmholtz equation to a simpler, approximate, quasi-optical, parabolic equation (PE). There are a multitude of physical phenomena which are not described within a PE framework, however. First, a PE entirely ignores back scattering. Second, even when back scattering is negligibly small, a wave can propagate in a large span of angles and the PE, with the small-angle approximation lying at its foundation, is inapplicable. Third, even if a wave propagates at small angles to the \( x \) axis and back scattering is unimportant, the Fresnel approximation can be violated over long enough paths and the PE becomes invalid.

Thus, a "dead zone" exists around the region of applicability of the PE and to fill it one must introduce approximate equations more general than the PE but simpler than the Helmholtz stochastic equation.

Such approximate equations, based on an expansion of the solution of the Helmholtz stochastic equation in a series by multiples of the back scattering, have been proposed in [1, 2]. In the present article we study some consequences of the equation obtained in [2] with back scattering neglected, which is also valid when the two conditions of applicability of the PE are violated: the Fresnel and small-angle approximations.

2. The equation for \( E(x, p) \) which follows from the Helmholtz stochastic equation (1) with back scattering neglected has the form [2]

\[ \frac{\partial E}{\partial x} = \mathcal{M}E + \frac{k^2}{2} \mathcal{N}zE, \]  \hspace{1cm} (2)

where \( \mathcal{M} \) and \( \mathcal{N} \) are the following integral operators:

\[ \mathcal{M} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\Lambda - k^2) \frac{e^{iR}}{R} dq, \]

\[ \mathcal{N} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{iR}}{R} dq, \quad R = \sqrt{(\rho - q)^2}. \]

In practical calculations it is often more convenient to use not Eq. (2) but an equation for

\[ E_x(x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} E(x, \rho) e^{-i(\rho \phi)} d\rho, \]

a Fourier transform of the field \( E(x, \rho) \) with respect to the transverse coordinates. The equation for \( E_x(x) \) corresponding to (2) is
\[ \frac{\partial E_\mathbf{x}}{\partial x} = i \sqrt{\kappa^2 - \kappa^2} E_\mathbf{x} + \frac{k^4}{2i \sqrt{\kappa^2 - \kappa^2}} \int_{-\infty}^{\infty} \Omega \, E_\mathbf{x} \, d\Omega \]  

(3)

We will assume that the function \( \varepsilon_\Omega(x) \) entering in here, the Fourier transform of \( \varepsilon(x, \rho) \), is Gaussian and has the following correlation function:

\[ \langle \varepsilon_\Omega(x) \varepsilon_\Omega(x + \xi) \rangle = \Phi_\varepsilon(\Omega) \delta(\Omega + \Omega') \delta(\xi). \]

(4)

Averaging Eq. (3) and closing the average \( \langle \varepsilon E_\mathbf{x} \varepsilon - \Omega \rangle \) in the diffusion approximation, we obtain

\[ \frac{\partial \langle E_\mathbf{x} \rangle}{\partial x} = \langle E_\mathbf{x} \rangle \left[ i \frac{k^4}{\sqrt{\kappa^2 - \kappa^2}} - \frac{k^4}{8 \sqrt{\kappa^2 - \kappa^2}} \int_{-\infty}^{\infty} \frac{\Phi_\varepsilon(\Omega)}{\sqrt{\kappa^2 - (\kappa - \Omega)^2}} d\Omega \right]. \]

The solution of this equation has the form

\[ \langle E_\mathbf{x}(x) \rangle = E_\mathbf{x}(0) \exp \left[ i \frac{k^4}{8 \sqrt{\kappa^2 - \kappa^2}} - \frac{k^4}{8 \sqrt{\kappa^2 - \kappa^2}} \int_{-\infty}^{\infty} \frac{\Phi_\varepsilon(\Omega)}{\sqrt{\kappa^2 - (\kappa - \Omega)^2}} d\Omega \right]. \]

(5)

Equation (5) allows us to draw certain conclusions about the limits of applicability of the quasi-optical parabolic equation in a statistical description of waves propagating in a randomly inhomogeneous medium. The first limitation of this kind is the limitation of the Fresnel approximation, i.e., the validity of the replacement of \( \sqrt{\kappa^2 - \kappa^2} \) by \( i k - \frac{x^2}{2k} \) in (13). Another limitation is directly connected with the influence of random inhomogeneities of the medium. Let us discuss it. Here we will assume for simplicity that the inhomogeneities \( \varepsilon(x, \rho) \) of the medium are isotropic and monoscalar with a characteristic scale \( l_0 \), while the characteristic scale of the field \( E_\mathbf{0}(\rho) \) is \( \rho_0 \). If \( l_0, \rho_0 > 2 \pi/k \), then the roots \( \sqrt{\kappa^2 - \kappa^2} \) and \( \sqrt{\kappa^2 - (\kappa - \Omega)^2} \) in (13) can be expanded in a Taylor series, and we can be confined in this case to terms proportional to \( \kappa^2 \) and \( \Omega^2 \). As a result, we obtain

\[ \langle E_\mathbf{x}(x) \rangle = E_\mathbf{x}(0) \exp \left[ i \frac{k^4}{8 \sqrt{\kappa^2 - \kappa^2}} - \frac{k^4}{8 \sqrt{\kappa^2 - \kappa^2}} \int_{-\infty}^{\infty} \frac{\Phi_\varepsilon(\Omega)}{\sqrt{\kappa^2 - (\kappa - \Omega)^2}} d\Omega \right]. \]

(6)

Here we introduce the notation

\[ A = A(0) = \int_{-\infty}^{\infty} \Phi_\varepsilon(\Omega) d\Omega, \]

\[ D = \frac{1}{4} \int_{-\infty}^{\infty} \Omega^2 \Phi_\varepsilon(\Omega) d\Omega. \]

The first three terms in the exponent of (6) correspond to the parabolic approximation while the last two describe the additional attenuation of the average field connected with the increase in the path traveled by the wave in an inhomogeneous medium, due to fluctuations in the angles of propagation of the waves and the oblique incidence of the individual plane waves making up the original wave \( E_\mathbf{0}(\rho) \). Within the framework of the small-angle approximation the term \( Dx/4 \) is much less than one and it can be discarded in comparison with \( k^2Ax/8 \). But the term \( \kappa^2Ax/8 \) is proportional to \( \sigma^2Ax_\rho/\rho_0^2 \) \( (\sigma^2) \), and if \( \rho_0 < l_0 \) it can lead to a considerable difference between the true average field and the average field within the framework of the parabolic approximation at distances \( x \sim \rho_0^2/\sigma^2l_0 \), at which the Fresnel and small-angle approximations can still be valid.

3. Let us consider corrections to the wave coherence function \( E(x, \rho) \) calculated within the framework of the quasi-optical parabolic equation. We define the wave coherence function \( E(x, \rho) \) by the equation

\[ \Gamma(x, s) = \frac{1}{ik} \langle E^*(x, \rho) E(x, \rho + s) \rangle. \]

Here and below the incident wave is assumed to be plane: \( E_\mathbf{0}(\rho) = 1 \). We also introduce the function

\[ S(x, x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \Gamma(x, s) e^{-i(\xi s)} d\xi. \]

It is easy to show that in the diffusion approximation \( S(x, \rho) \) satisfies the following equation:

\[ \frac{\partial S}{\partial x} = -\frac{k^4}{4 \sqrt{\kappa^2 - \kappa^2}} \left[ S + i \frac{\Phi_\varepsilon(\Omega)}{\sqrt{\kappa^2 - (\kappa - \Omega)^2}} d\Omega - \int_{-\infty}^{\infty} \frac{\Phi_\varepsilon(\Omega)}{\sqrt{\kappa^2 - (\kappa - \Omega)^2}} d\Omega \right]. \]

(7)