ASYMPTOTIC BEHAVIOR OF THE THERMAL
VIBRATIONS OF A DEFORMED CRYSTAL LATTICE

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The asymptotic behavior of the dynamical equations for the displacements of the atoms of a crystal lattice with respect to the unit-cell parameter \( a \) (the maximal length of vectors of unit translations) is studied for a given external long-wave deformation. In the long-wave limit the equations go over into those of elasticity theory with variable coefficients. The explicit form of the asymptotic solution is given and some concrete cases of external deformation — homogeneous and in the form of a plane traveling wave — are analyzed.

The development of experimental techniques in recent years has greatly stimulated interest in analyzing the passage of ultrasound through a crystal [1-3]. At the same time theoretical studies have been made, these being mainly devoted to a semiphenomenological analysis of the different interactions of phonons from the point of view of kinetic equations [4-7].

The present paper is devoted to some aspects of the dynamical behavior of thermal phonons of an ideal nonionic dielectric crystal for a given external deformation. In particular, the case when the external deformation is long-wave sound is analyzed.

1. Equation of Motion and Statement of the Cauchy Problem

Suppose that the equilibrium position of atom \( \chi \) of unit cell \( l \) is specified by the vector (the notation largely corresponds to that in [8])

\[
x^{(l)}(\chi) = x(l) + x(\chi) = \sum_{m=1}^{s} l_m a_m + x(\chi),
\]

where \( l_m \) are integers, the set of them being denoted by \( l \), and \( a_m \) are noncoplanar vectors of unit translations. The deviation of an atom from the equilibrium position will be denoted in Cartesian coordinates by a sum of two vectors:

\[
u^{(l)}(\chi,t) + \xi(x^{(l)}(\chi),t),
\]

where the first denotes the displacement of the atom due to the thermal fluctuations and the second characterizes the induced motion of the atom under the influence of external fields.

In what follows, the investigated quantity will be \( u^x(l,t) \), and the treatment will be given in the harmonic approximation for the potential energy \( U \):

\[
U(..., x+\xi_1 + u_1, ...) \approx U(..., x+\xi_1, ...) + \sum_{l=1}^{s} \frac{\partial U}{\partial u^x(l,t)} \bigg|_{u=0} u^x(l,t) + ...
\]
The equation for motion for the thermal displacements is then

$$M \ddot{u}(t) + \sum_{j \neq \beta} \frac{\partial^2 U}{\partial u_j(t) \partial u_{\beta}'(t')} \mid_{u_{\beta}=0} \dot{u}_{\beta}'(t') + \frac{\partial U}{\partial u_j(t)} \mid_{u_{\beta}=0} + M \dddot{u}_j(t) = 0.$$  

(1.4)

We are interested in this problem. Suppose that $t=0 \zeta(x(t), t) = 0$ up to the time $t=0$. Then the general solution of (1.4) can be written in the form [8].

$$u^s(l, t) = \frac{1}{i\sqrt{\lambda M}} \text{Re} \sum_{j=1}^{\infty} \sum_{k} e_{s}^{j}(j(k)) \sqrt{\frac{2\hbar}{\omega_{j}(k)}} Q(j, k) e^{i(j(k)-s, j(k))}.$$  

(1.5)

We introduce $a = \max |\alpha_m|$, which will play the role of a small parameter in what follows. Suppose at $t=0$ a forced motion $\zeta_{\lfloor t \rfloor}$ of the atoms of the crystal is slowly "switched on". We shall seek the solution of the Cauchy problem for Eq. (1.4) defined for $t \leq 0$ by Eq. (1.5) in the limit $a \to 0$.

To solve the problem we make the following assumptions.

1. The interaction has short-range order (short-range interaction) and the region of appreciable variation of $\zeta_{\lfloor t \rfloor}$ is much greater than the interaction region (long-wave deformation). In this case, the difference of the coordinates between the "perturbed" atoms whose equilibrium position is determined by the vectors $x_{\lfloor t \rfloor}$ and $x_{\lfloor t' \rfloor}$ subject to the condition that $|l-l'|$ is less than the interaction range can be written in the form

$$r_{s}(l, l') = x_{s}(l, t) + \zeta_{s}(x_{\lfloor t \rfloor}, t) - x_{s}(l', t) - \zeta_{s}(x_{\lfloor t' \rfloor}, t) \approx$$

(1.6)

$$\approx \sum_{p=1}^{3} \left[ x_{s}(l, \chi') - x_{s}(l', \chi') \right] \left[ \delta_{s}^{\mu} + \frac{\partial}{\partial x_{s}(l, \chi)} \zeta_{s}(x_{\lfloor t \rfloor}, t) \right].$$

For us it is now important that $r_{s}(l, l')$ depends linearly on $l-l'$ and arbitrarily on $l$ but not arbitrarily on $l$ and $l'$, as would be in the general case.

2. We make a further important assumption associated with estimating the quantities in (1.4); the meaning of it will be clarified in Sec. 2. Namely, with allowance for assumption 1, that

$$\left. \frac{\partial^2 U}{\partial u_{s}(l, t) \partial u_{s}(l', t')} \right|_{u_{s}=0} = \frac{1}{a^r} \Phi_{s s}(l-l', \chi, \chi') \frac{\partial^2}{\partial x_{s}(l, \chi)}.$$  

(1.7)