When is a Stochastic Integral a Time Change of a Diffusion?

Bernt Øksendal

Received August 8, 1988; revised November 22, 1989

We give necessary and sufficient conditions that a time change of an n-dimensional Ito stochastic integral $X_t$ of the form

$$dX_t = u(t, \omega) \, dt + v(t, \omega) \, dB_t,$$

leads to a process with the same law as a diffusion $Y_t$ of the form

$$dY_t = b(Y_t) \, dt + \sigma(Y_t) \, dB_t,$$

where the generator $A$ of $Y_t$ is assumed to have a unique solution of the martingale problem. The result has applications to conformal martingales in $\mathbb{C}^n$ and harmonic morphisms.

KEY WORDS: Stochastic integral; diffusion; conformal martingales.

1. INTRODUCTION AND STATEMENT OF RESULTS

In the following we will let $X_t = X_t^\omega$ denote an Ito stochastic integral in $\mathbb{R}^n$

$$dX_t = u(t, \omega) \, dt + v(t, \omega) \, dB_t, \quad X_0 = x \in \mathbb{R}^n \quad (1.1)$$

where $B_t = (B_t, \Omega, \mathcal{F}_t, P^\omega)$ is an $n$-dimensional $\mathcal{F}_t$-Brownian motion and $u(t, \omega) \in \mathbb{R}^{n \times 1}$, $v(t, \omega) \in \mathbb{R}^{n \times n}$ are processes satisfying certain conditions that we will specify later. And we let $Y_t = Y_t^\omega(\omega)$ denote an Ito diffusion, i.e., a (weak) solution of an Ito stochastic differential equation (in $\mathbb{R}^n$)

$$dY_t = b(Y_t) \, dt + \sigma(Y_t) \, dB_t, \quad Y_0 = x \in \mathbb{R}^n \quad (1.2)$$

The conditions on $b$ and $\sigma$ will also be specified later.

1 Department of Mathematics University of Oslo Box 1053, Blindern N-0316 Oslo 3 Norway.
Next we describe the time changes that we will consider: Let $c(t, \omega) \geq 0$ be an $\mathcal{F}_t$-progressively measurable process \{i.e., $(s, \omega) \mapsto c(s, \omega)$ of $[0, t] \times \Omega \to \mathbb{R}$ is measurable for all $t$ with respect to $\mathcal{B}[0, t] \times \mathcal{F}_t$, where $\mathcal{B}$ denotes the Borel $\sigma$-algebra\}. Define

$$\beta_t = \beta(t, \omega) = \int_0^t c(s, \omega) \, ds$$

(1.3)

and let $\alpha_t$ be the right continuous right inverse of $\beta_t$:

$$\alpha_t = \inf\{s; \beta_s > t\} \quad (\alpha_t = \infty \text{ if } \beta_s \leq t \text{ for all } s)$$

(1.4)

Then we say that $\alpha_t$ is a time change with time change rate $c(t, \omega)$. Note that $\omega \mapsto \alpha(t, \omega)$ is an $\{\mathcal{F}_t\}$-stopping time for each $t$, since $\{\omega; \alpha(t, \omega) < s\} = \{\omega; t < \beta(s, \omega)\} \in \mathcal{F}_t$. We now ask the question:

When is the time changed process $X_{\alpha}$, identical in law with $Y_t$ (in short: $X_{\alpha} \sim Y_t$)?

(1.5)

The motivation for this problem comes from several situations. We mention three of them:

1.1. Filtering Theory

Suppose the observation process $H_t$ in $\mathbb{R}^n$ is of the form

$$dH_t = z(t, \omega) \, dt + dB_t$$

Let $\mathcal{N}_t$ denote the $\sigma$-algebra generated by $\{H_s; s \leq t\}$. Then the innovation process $X_t$ is given by

$$dX_t = \{z_t - E[z(t, \cdot) | \mathcal{N}_t]\} \, dt + dB_t$$

(1.6)

For more information about filtering theory see Ref. 6. See also Ref. 11, which includes a detailed calculation of $E[z(t, \cdot) | \mathcal{N}_t]$ (the Kallianpur–Striebel formula).

It is an important fact that $X_t$ is a Brownian motion w.r.t. $\mathcal{N}_t$. (See, e.g., Ref. 9, Theorem 7.12.) This raises the more general question: When is a stochastic integral

$$dX_t = u(t, \omega) \, dt + v(t, \omega) \, dB_t$$

a Brownian motion? The situation above describes a sufficient condition in the case when $v$ is the identity matrix. In the case when $u = 0$ and $n = 1$ it is a direct consequence of Levy's martingale characterization of Brownian