Complete Convergence of Weighted Sums of Martingale Differences

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Let $F_0 \subset F_1 \subset \cdots$ be an increasing family of $\sigma$-algebras. For each $n \geq 1$, $X_n$ is $F_n$ measurable, and $E(X_n|F_{n-1})$ is zero almost surely, and $E(|E_n|^p|F_{n-1})$ is bounded by a finite constant almost surely for some $p \geq 2$. Let $a_{n1}, \ldots, a_{nn}$ be constants. Conditions are given to establish the complete convergence of 

$$\frac{a_{n1}X_1 + \cdots + a_{nn}X_n}{n^{1/p}},$$

thereby obtaining an extension of Chow's (1966) result for the case of independent and identically distributed random variables. When $p > 2$, the conditions are an improvement on existing results for the case of independence and identical distribution.

KEY WORDS: Complete convergence; weighted sums; Martingale differences.

1. INTRODUCTION

Chow\(^{(1)}\) has established the following complete convergence result.

**Theorem 1** (Chow\(^{(1)}\)). Let $X, X_1, \ldots$ be independent and identically distributed random variables with $EX = 0$ and $E|X|^p < \infty$ for some $p \geq 2$. Let $a_{n1}, \ldots, a_{nn}$ be constants for each $n \geq 1$ and

$$U_n = \sum_{i=1}^{n} a_{ni}X_i/n^{1/p}$$

If for some finite constant $K$ not depending on $n$

$$a_{n1}^2 + \cdots + a_{nn}^2 \leq K \quad (1.1)$$

and

$$n^{1/p} \max_{1 \leq i \leq n} |a_{ni}| \leq K \quad (1.2)$$

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then for all $\varepsilon > 0$

$$\sum_{i=1}^{\infty} P[|U_n| \geq \varepsilon] < \infty$$

In the same setting as Theorem 1, Thrum\(^{(4)}\) has obtained the almost sure convergence of $U_n$ to zero as $n$ tends to infinity under the mere assumption of the condition (1.1), that is, without assuming the condition (1.2). For the case when $0 < p \leq 2$, Teicher\(^{(3)}\) has obtained the almost sure convergence of $U_n$ to zero under the condition that $(\log n) \max_{1 \leq i \leq n} |a_{ni}| \leq K$. This overlaps with Thrum's theorem when $p = 2$.

Although the almost sure convergence result is interesting in its own right, it is by no means an improvement on Theorem 1. In this note, we shall attempt to strengthen Theorem 1 in two directions. We shall do away with the independence and identical distributions for the random variables $X_n$, and in their place we shall assume some martingale structure. In the other direction we shall deal with the assumptions on the constants $a_{ni}$. Specifically we shall prove the following theorem.

**Theorem 2.** Let $F_0 \subset F_1 \subset \cdots$ be an increasing family of $\sigma$-algebras and $\{(X_n, F_n), n \geq 1\}$ be a sequence of martingale differences, that is, for each $n \geq 1$

$$E(X_n | F_{n-1}) = 0, \text{ almost surely}$$

and $X_n$ is $F_n$-measurable. Assume that for some $p \geq 2$

$$E(|X_n|^p | F_{n-1}) \leq K, \text{ almost surely}$$

where $K$ is a constant not depending on $n$. For each $n \geq 1$, let $a_{n1}, \ldots, a_{nn}$ be constants and

$$U_n = \sum_{i=1}^{n} a_{ni} X_i / n^{1/p}$$

Assume for some $\delta < 1/p$

$$a_{n1}^2 + \cdots + a_{nn}^2 \leq Kn^\delta \quad (1.3)$$

and

$$\sum_{i=1}^{\infty} \left( \sum_{i=1}^{n} |a_{ni}|^p \right) / n < \infty \quad \text{if} \quad 2 \leq p < 4$$

$$\sum_{i=1}^{\infty} \left( \sum_{i=1}^{n} a_{ni}^p \right) / n < \infty \quad \text{if} \quad p \geq 4 \quad (1.4)$$