ELEMENTARY EXTREMAL SYSTEMS OF THE SELF-OSCILLATORY TYPE WITH AN INERTIAL CONTROLLED ELEMENT

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An elementary extremal system of the self-oscillatory type with a second-order inertial controlled element is discussed. The usual optimizer, which bases the control signal on analysis of the signs of the input and output first derivatives of the controlled element, is shown to give unsatisfactory results with this system. Ways are suggested for avoiding these drawbacks.

Fig. 1

1. Elementary extremal systems of the self-oscillatory type (with one input and one output) with a member consisting of a nonlinear inertialess element in series with a linear first-order inertial element have been discussed in a number of papers (e.g., [1-3]). Various defects in the operation of the control system have been thereby revealed, leading either to instability or to a poor process of search for the output extremum. It has been discovered that one of the main causes of instability is the existence in the system phase space of regions of sliding motion of the same dimensions as the phase space itself.

In the present paper we consider an elementary extremal system of the self-oscillatory type with a controlled system consisting of a nonlinear inertialess element in series with a linear second-order inertial element. A block diagram of the system is shown in Fig. 1; its equations of motion are

\[ \ddot{u} + 2h \dot{u} + \varphi = -u^2, \quad \dot{u} = \varphi(t-\theta), \quad \eta = \Phi[\dot{u}, \varphi]. \]  

(1)

where \( h \) and \( \Delta \) are positive parameters, and \( \theta \) is a parameter characterizing the delay in the optimizer. The nonlinear function \( \Phi[\dot{u}, \varphi] \) can be realized by two polarized relays (Fig. 2).

Fig. 2

We show in the present paper that the phase space of system (1) also contains a region of sliding motion of the same dimensions as the phase space itself.

This leads here, not only to possible search instability, but also to a substantial increase in the output extremum tracking error. These defects are inherent in all extremal systems containing an inertial element of higher order than the first. The system discussed here is the simplest type in this class.

Ways of avoiding these drawbacks will be considered.

2. For \( \theta = 0 \) the phase space of system (1) is three-dimensional \((u, \varphi, \dot{\varphi} = \phi)\), generated by two subspaces \( \Phi_{+1} \) and \( \Phi_{-1} \) such \( \eta \) is equal to +1 and -1, respectively, in them. This phase space is the three-dimensional analog of the two-sheeted plane phase space [1-3].

We consider the space \( \Phi_{+1} \). System (1) has no equilibrium states in this space. According to (1), the phase point moves in \( \Phi_{+1} \), till the condition \( \dot{\varphi} + \Delta \Delta = 0 \) is violated. If the phase point reaches the region \( \varphi = -\Delta \) of space \( \Phi_{+1} \), it moves immediately into the space \( \Phi_{-1} \). We denote the plane \( \varphi = -\Delta \) by II. This plane divides \( \Phi_{+1} \) into two parts \( \Phi_{+1}^+ \), where \( \varphi > -\Delta \), and \( \Phi_{+1}^- \), where \( \varphi < -\Delta \).

Fig. 3

It follows from the first equation of system (1) that \( \dot{\varphi} = -\varphi - 2h \varphi - u^2 \). Hence \( \dot{\varphi} \) vanishes on the surface \( \varphi + 2h \varphi + u^2 = 0 \), which we denote by \( F \). The region \( \Phi_{+1}^+ \) divides this surface into two parts: \( \Phi_{+1}^+ \), where \( \dot{\varphi} > 0 \), and \( \Phi_{+1}^- \), where \( \dot{\varphi} < 0 \). The surface \( F \) cuts the plane II along the line \( L(\varphi = -\Delta, \varphi = 2h\Delta - u^2) \), and divides it into two parts: \( \Pi^+ \), where \( \dot{\varphi} > 0 \), and \( \Pi^- \), where \( \dot{\varphi} < 0 \). The phase point travels from the region \( \Pi^+ \) into the region \( \Phi_{+1}^+ \), and from \( \Pi^- \) into \( \Phi_{+1}^- \).

Differentiating the first equation of system (1) with respect to \( t \) and substituting the expression for \( \dot{\varphi} \) in the resulting equation, we find that, with \( \eta = +1 \), \( \dot{\varphi} = 2h \varphi - \varphi (1 - 4h^2) + 2hu^2 - 2u \). The surface \( \Phi_{+1}^+ \) on which \( \dot{\varphi} = 0 \) cuts the surface \( F \) and divides it into two parts: \( F^+ \), where \( \dot{\varphi} > 0 \), through which the phase trajectories enter \( \Phi_{+1}^+ \) from \( \Phi_{+1}^- \), and \( F^- \), where \( \dot{\varphi} < 0 \), through which the phase trajectories depart from \( \Phi_{+1}^+ \) into \( \Phi_{+1}^- \).
The surface $\psi_{+1}$ cuts the line $L$ at the point
\[ A \left( \frac{\Delta}{2}, 2\Delta - \frac{\Delta^2}{4}, -\Delta \right). \]

Thus the phase point, having started its movement from the region $\Pi^+$ of space $\Phi_{+1}$, enters the region $\Phi_{+1}^+$, passes through $\Phi_{+1}^{++}$ and $\Phi_{+1}^{+++}$, and enters the region $\Pi^+$, from which it departs into $\Phi_{-1}$ (Fig. 3).

It follows from the form of system (I) that the phase trajectories of space $\Phi_{-1}$ are symmetrical with the phase trajectories of space $\Phi_{+1}$ relative to the plane $u = 0$. The equations of the plane $\Pi$ and surface $F$ are independent of $\eta$. The plane and surface therefore remain unchanged when passing from space $\Phi_{+1}$ to space $\Phi_{-1}$; the plane II divides space $\Phi_{+1}$ into two parts; $\Phi_{-1}^+$ and $\Phi_{-1}^-$, where $\psi > 0$, and $\Phi_{-1}^+$, where $\psi < 0$. If the phase point starts its movement in the region $\Pi^+$ of space $\Phi_{+1}$, it enters the region $\Phi_{+1}^+$ passes through $\Phi_{+1}^{++}$ and $\Phi_{+1}^{+++}$ and enters $\Pi^+$, from which it departs into $\Phi_{-1}$.

We have described above the phase point motion in the part of the phase space where $\phi > -\Delta$. In the part where $\phi < -\Delta$, the phase point cannot travel only in space $\Phi_{+1}$, or only in $\Phi_{-1}$, but moves constantly back and forth between $\Phi_{+1}$ and $\Phi_{-1}$. For, from the equations of motion, having entered space $\Phi_{+1}$ (or $\Phi_{-1}$), the phase point must immediately pass into space $\Phi_{-1}$ (or $\Phi_{+1}$ respectively). In other words, the phase point has a sliding motion along the trajectory anywhere in the part of the phase space where $\phi < -\Delta$. The sliding mode trajectories thus fill a three-dimensional region of the phase space.

If the phase point passes from the region $\phi > -\Delta$ to the region $\phi < -\Delta$ of the phase space, and cuts the plane II at a point with coordinate $u = u_1$, it will move along the sliding mode trajectory in accordance with the equation $\phi + 2\Delta \xi + \psi = -u_1$. For, let the relays $P_1$ and $P_2$ (Fig. 2) have delays $\theta_1$ and $\theta_2$, and let the phase point, as it moves along a trajectory of space $\Phi_{+1}$ (i.e., $\Phi_{+1}$ or $\Phi_{-1}$) cut the region $\Pi^+$ of the plane at $t = 0$ in the point with coordinate $u = u_1$. After cutting the plane $\Pi^+$, the phase point moves for $\theta_1$ units of time along the trajectory of space $\Phi_{+1}$, then the space $\Phi_{-1}$, here, $u(\theta_1) = u_1 + \varphi_1$. The phase point moves then along a trajectory of space $\Phi_{-1}$ for $\theta_2$ units of time ($\phi = -\theta_1 + \theta_2$), after which it again passes into space $\Phi_{+1}$; here, $u(\theta_1 + \theta_2) = u(\theta_1) - \phi_2 = u_1 - \phi_2$. After this, the phase point moves for $\theta_2$ units of time in space $\Phi_{+1}$, and so on till it again reaches the plane II. Here, $u(\theta_2 + 2\Delta \xi + \psi) = u_1 - \phi_1$, and $u(\theta_2 + 2\Delta \xi) = u_1 + \phi_2$, ($n = 0, 1, ...$). If $\theta_1$ and $\theta_2$ tend to zero, we find in the limit that, for the sliding mode, $u(t) = u_1$ everywhere in the region $\phi < -\Delta$ of the phase space.

It can easily be shown that the phase point enters the region $\Pi^-$ of the plane II along a sliding mode trajectory. It is here that the question arises: does it continue along a trajectory of space $\Phi_{+1}$, or of $\Phi_{-1}$. With $\theta_1 = \theta_2 = 0$ there can be no definite answer to this question, since, after it has cut the region $\Pi^-$, there is a certain probability of the phase point moving along a trajectory in either of these spaces; $\Phi_{+1}$ or $\Phi_{-1}$.

If the delays $\theta_1$ and $\theta_2$ are nonzero, a definite answer can be given. It was shown above, in fact, that with $\phi < -\Delta$ and $\theta > 0$, the phase point moves as follows: if, when it cuts the region $\Pi^-$, it was moving along a trajectory of space $\Phi_{+1}$, then $\theta_1$ units of time after cutting $\Pi^-$ it will be moving along a trajectory of space $\Phi_{+1}$, for $\theta$ units it will move along a trajectory of space $\Phi_{-1}$, for $\theta$ units again along a trajectory of $\Phi_{+1}$, and so on. Thus the phase point reaches the region $\Pi^+$ in this case along a trajectory of each space $\Phi_{+1}$ or $\Phi_{-1}$. If it was moving along a trajectory of space $\Phi_{+1}$, for a period not exceeding $\theta$ before intersecting the region $\Pi^+$, after intersecting $\Pi^+$, it will move along a trajectory of space $\Phi_{-1}$.

The time taken by the phase point to travel from the region $\Pi^-$ to $\Pi^+$ depends largely on its initial position in $\Pi^-$. It can be shown that the entire region $\Pi^-$ is divided up by nonintersecting lines, along each of which this time is constant. Similarly, $\Pi^+$ is divided up by lines, for each of which the time taken by the phase point to travel from $\Pi^-$ to $\Pi^+$ is constant. The lines of the region $\Pi^-$ for which the time is equal to $2n \theta + \theta_1$, and $2n \theta + \theta_2 + \theta_2$ ($n = 0, 1, ...$), divide $\Pi^-$ into a denumerable set of subregions. We denote by $\Phi_{-1}$ the subregions for which the phase point traveling time from $\Pi^-$ to $\Pi^+$ is $t \in (0, \theta_1)$, and by $\Phi_{+1}$ ($n = 1, 2, ...$) the subregions for which the time $t \in (2n \theta - n + \theta_1, 2n \theta + \theta_2)$, while $\beta_n$ ($n = 0, 1, 2, ...$) are the subregions for which $t \in (2n \theta + \theta_1, 2n \theta + \theta_2 + \theta_2)$.

If the phase point, after intersecting $\Pi^+$, moves along a trajectory of space $\Phi_{+1}$, it will reach the subregions $\Phi_{+1}$ along trajectories of space $\Phi_{-1}$, and the $\Phi_{-1}$ along trajectories of space $\Phi_{+1}$; further, it will move into the region $\Pi^-$ along trajectories of space $\Phi_{+1}$ from the $\Phi_{-1}$, and along trajectories of space $\Phi_{-1}$ from the $\Phi_{+1}$.

It can be shown that the distance along the $\phi$ axis between the boundaries of each of the regions $\Phi_{+1}$ and $\Phi_{-1}$ ($n = 0, 1, ..., \theta$) tends to zero as $\theta \to 0$, i.e., these regions degenerate into lines as $\theta \to 0$. Hence follows our above statement that, with $\theta = 0$, no definite answer can be given regarding the behavior of the phase point after it has cut the region $\Pi^+$. In actual equipment the delays $\theta_1$ and $\theta_2$ are nonzero. Hence the region $n^+$, $n > 0$, is divided into subregions $\Phi_{+1}$ and $\Phi_{-1}$ ($n = 0, 1, ...$). If $\theta_1$ and $\theta_2$ are small, the distances along the $\phi$ axis between the boundaries of these subregions will also be small. Hence the phase space structure is quite complex in this case, and it is not possible to trace the behavior of the phase point over any substantial period of time. In particular, nothing definite can be said about the existence, nature and stability of the periodic modes of the system. All we can reasonably say is that the modes will be complex and that the system will have low noise immunity because $\Pi^+$ is split up into alternating subregions $\Phi_{+1}$ and $\Phi_{-1}$. For, let a closed phase trajectory $\Gamma$ corresponding to a stable periodic mode pass through $\Phi_{+1}$, and let the system by affected by noise which does not take