Generalized One-Sided Laws of the Iterated Logarithm for Random Variables Barely with or without Finite Mean

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The almost sure limiting behavior of weighted sums of independent and identically distributed random variables barely with or without finite mean are established. Results for these partial sums,

$$\sum_{k=1}^{n} k^{\alpha}X_k, \quad \alpha \in \mathbb{R}$$

have been studied, but only when $\alpha = -1$ or $\alpha = 0$. As it turns out, the two cases of major interest are $\alpha = -1$ and $\alpha > -1$. The purpose of this article is to examine the latter.

KEY WORDS: Law of the iterated logarithm; strong law of large numbers; slow variation; weak law of large numbers.

1. INTRODUCTION

Throughout, $\{X, X_n, n \geq 1\}$ will be i.i.d. unbounded asymmetrical random variables with larger right tail than left. We will explore the almost sure limiting behavior of weighted sums of these random variables when strong laws fail. In Adler and Rosalsky (1) the authors showed that if $\{X, X_n, n \geq 1\}$ are i.i.d. nonintegrable random variables and $\{a_n, n \geq 1\}$ are constants satisfying

$$\sum_{k=1}^{n} |a_k| = O(n |a_n|) \text{ and } n |a_n| \uparrow$$

(1.1)

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then

\[ P \left\{ \lim_{n \to \infty} \sum_{k=1}^{n} a_k X_k / b_n = 1 \right\} = 0 \]

for all sequences \( \{ b_n, n \geq 1 \} \).

On the other hand if (1.1) fails, then a strong law can exist; see Adler.\(^2\) The important point here is that those random variables are barely with or without finite means. Here we unite these two ideas. This is accomplished by examining weights where (1.1) holds and random variables of the type that can be found in Adler\(^2\); see Sec. 4 of this paper.

These interesting random variables were also studied by Klass and Teicher.\(^3\) Many of the results obtained here owe much to the work of these two men. Furthermore, some of the properties used freely here can be found in Klass and Teicher.\(^3\) This allows us to omit details at times.

A few remarks about notation are needed. The symbol \( C \) will denote a generic finite constant which is not necessarily the same in each appearance. Also, let \( \log x = \log_e(\max \{ e, x \} ) \) and \( \log_2 x = \log \log x \).

2. PRELIMINARIES

As in Klass and Teicher\(^3\) let

\[ \tilde{\mu}(x) = \int_{-\infty}^{\infty} P\{|X| > t\} \, dt \quad \text{provided} \quad E|X| < \infty \]

and

\[ \mu(x) = \int_{0}^{x} P\{|X| > t\} \, dt \quad \text{when} \quad E|X| = \infty \]

Also, set

\[ c_x = \left( \frac{x}{\tilde{\mu}(x)} \right)^{-1} \quad \text{when} \quad E|X| < \infty \]

and

\[ c_x = \left( \frac{x}{\mu(x)} \right)^{-1} \quad \text{when} \quad E|X| = \infty \]

Thus, \( c_n = n\tilde{\mu}(c_n) \) if \( E|X| < \infty \) and \( c_n = n\mu(c_n) \) if \( E|X| = \infty \).

We consider weights of the form \( a_n = n^z, n \geq 1 \). In Adler\(^2\) we studied the behavior of our partial sum, \( S_n = \sum_{k=1}^{n} a_k X_k, n \geq 1 \), when \( z = -1 \). The