Hopf Bifurcation for Functional Differential Equations of Mixed Type

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We prove Hopf bifurcation and center manifold theorems for functional differential equations of mixed type. An application to the dynamic behavior of a competitive economy (business cycle) is provided.

KEY WORDS: Hopf bifurcation; center manifold; periodic solutions; mixed functional differential equations.

1. INTRODUCTION

Mixed functional differential equations (MFDE) are here a special class of functional differential equations where the time derivative depends on both past and future values of the variable. The reader is referred to Rustichini (1989) for a brief discussion of the motivations for the study of such equations.

In this paper, we deal with two aspects of the theory of nonlinear MFDEs: Hopf bifurcation and the center manifold theorem. In a final section, we present an application of this theory to a problem arising in economic theory (existence of business cycles in a competitive economy).

We consider first the Hopf bifurcation. We recall that, broadly speaking, Hopf bifurcation theorems prove the existence of periodic solutions of a nonlinear equation, in the vicinity of a stationary solution, when a conjugate pair of distinct eigenvalues of the linearized equation crosses the imaginary axis. In the proof of the Hopf bifurcation theorem for MFDE, a strategy of proof is necessary that does not involve a solution operator. The argument that we adopt is a purely functional analytic one, involving a Lyapunov–Schmidt reduction (LSR). We set our problem in

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the space of periodic functions of fixed period. The linearization of the stationary solution of our MFDE defines a linear operator acting on this space. The key step, in order to set the LSR, is the proof that the linear operator so defined is a Fredholm operator. This difficulty can be overcome thanks to our choice of the space: a linear operator of mixed type, when its action is restricted to the periodic functions, can be in fact identified with an operator of the delay type. Once this is done, the task is reduced to the study of the zeros of the bifurcation functions.

In Section 5, we deal with the existence of a center manifold. Let us recall briefly the main idea of the center manifold theorem (for ordinary, or delay, equations). One again considers the behavior of a nonlinear equation in the vicinity of a stationary solution; if the characteristic equation of the linearized equation at the stationary solution, has, say, a pair of characteristic roots on the imaginary axis, then this same linear equation has a two-dimensional (2D) subspace of solutions that have exponentially bounded growth.

It is natural to ask whether the nonlinear problem has, at least locally, a 2D submanifold of solutions, homeomorphic to the 2D subspace. The affirmative answer is given in the center manifold theorem. It is important to emphasize that the linear subspace and the submanifold mentioned are both defined in the phase space; the center manifold is locally the graph of a function from (a subspace of) the space of continuous functions on the delay interval into itself. This is very natural in a situation where a continuous semigroup is defined, and the map defining the center manifold is found by means of a variations of constants formula built upon this semigroup [see, for instance, Hale (1977)].

In the case of MFDE, this is no longer possible due to the lack of such continuous semigroups [see Rustichini (1989)]. Therefore, a different strategy is necessary, similar in spirit to the Lyapunov–Schmidt decomposition. First of all, it involves the choice of a space different from the (continuous) functions on the interval $[-r, r]$. We consider the space of continuously differentiable functions defined over the entire real line, with norm weighted by an exponential factor. The linear operator, $M$, defined by the linearization of the MFDE, and the MFDE itself both naturally define a map on this space. The image space is the space of continuous functions defined over the entire real line, with a weighted norm. Once we prove that the dimension of the kernel of $M$ is the same as the number of roots (considering multiplicity) located on the imaginary axis, and that $M$ is surjective, then the implicit function theorem will provide the homeomorphism defining the center manifold. It should be emphasized that our center manifold theorem relies on a fairly strong condition on the linearized operator. We now introduce some notation used in the following.