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FREGE AND THE RIGORIZATION OF ANALYSIS

ABSTRACT. This paper has three goals: (i) to show that the foundational program begun in the *Begriffsschrift*, and carried forward in the *Grundlagen*, represented Frege’s attempt to establish the autonomy of arithmetic from geometry and kinematics; the cogency and coherence of ‘intuitive’ reasoning were not in question. (ii) To place Frege’s logicism in the context of the nineteenth century tradition in mathematical analysis, and, in particular, to show how the modern concept of a function made it possible for Frege to pursue the goal of autonomy within the framework of the system of second-order logic of the *Begriffsschrift*. (iii) To address certain criticisms of Frege by Parsons and Boolos, and thereby to clarify what was and was not achieved by the development, in Part III of the *Begriffsschrift*, of a fragment of the theory of relations.

1. INTRODUCTION

Paul Benacerraf, in his important paper, ‘Frege: the last logicist,’⁰ raises a number of questions about the intellectual motivations for Frege’s logicism. Benacerraf makes the correct point that when Frege’s foundational interests are viewed in their mathematical context, they stand in sharp contrast with the logical empiricists’ attempts to show the analyticity of arithmetic and more generally, of all *a priori* knowledge. Any applications of Frege’s logicism to such an empiricist theory of the *a priori* is at best an unintended by-product of his foundational concerns. Benacerraf argues that assimilating Frege to a much later empiricism distorts his purpose to such an extent that if this were the point of logicism, we should not look to Frege for its intellectual origins. Frege’s logicism comes at the end of another tradition, and that tradition is primarily a mathematical one. On Benacerraf’s account, Frege’s work is the culmination of the process of making rigorous the calculus and the theory of the reals. Frege sought to do for arithmetic what Cauchy, Bolzano, Weierstrass, Cantor and Dedekind did for analysis: viz., to secure for it a ‘rigorous foundation.’ But what exactly was the goal of rigorization?

For Benacerraf, Frege's interest in rigor is driven by the problem of providing a proper justification for believing the truth of the propositions of arithmetic: 'The propositions of arithmetic stand in need of proof' (p. 23) is Benacerraf's gloss on Frege's remark that '... in mathematics a mere moral conviction, supported by a mass of successful applications, is not good enough.' (Grundlagen 2 sect. 1, quoted by Benacerraf, p. 23.) According to Benacerraf, both Frege and the nineteenth century analysts viewed the process of rigorization as one of providing clear definitions and mathematical proofs where none had previously been given. This comports well with a number of Frege's statements, for example '... it is in the nature of mathematics always to prefer proofs, where proofs are possible...' (Grundlagen sect. 2, quoted by Benacerraf, p. 23), and '... the rigor of the proof remains an illusion... so long as the definitions are justified only as an afterthought, by our failing to come across any contradiction.' (Grundlagen p. xxiv, quoted by Benacerraf, pp. 22f.).

If Benacerraf's account of the goal of rigorization is accepted, then Frege's foundational investigations are epistemological in the standard sense that the questions and metaquestions they address surround the justification of mathematical propositions for the purpose of insuring that we will not 'in the end encounter a contradiction which brings the whole edifice down in ruins.' (Grundlagen p. xxiv, quoted by Benacerraf, p. 23.) It should therefore be clear that the question Benacerraf is raising is not whether Frege's foundational interests were mathematical or philosophical. They were obviously both. Benacerraf is claiming that our understanding of Frege's foundational interests is invariably distorted when these interests are viewed from the perspective of any particular philosophical school; although I don't address it directly, I believe that what I have to say is supportive of this broad interpretive claim.

It would be foolish to deny that the goals of cogency and consistency were an important part of the nineteenth century interest in rigor. And it is easy to find quotations from Frege which show that he sometimes at least wrote as if he took his task to include securing arithmetic, in the broad sense, which includes both the arithmetic of natural numbers and the theory of the reals, against contradiction. But I think it can be questioned just how far skeptical worries about the consistency or