INTRODUCTION

The instability of a liquid surface in an electric field, expressed in the formation of conical projections from which erupt jets that break up into small charged droplets, has long attracted the attention of investigators in connection with the problem of charge separation in thunder clouds [1]. This instability was observed as early as the 18th century [2], and the first theoretical analysis of the instability of the charged drop was carried out by Rayleigh at the end of the last century [3]. Despite the hundreds of experimental and theoretical studies devoted to this question, the nature of the development of the instability is still not clear.

According to the ideas advanced in [3], which are usually transferable from a charged drop to a drop in a field, the instability of a liquid drop in an external electric field is associated with the instability of the higher capillary wave modes. However, as noted in [4], exciting the higher capillary wave modes in a drop in accordance with the model proposed in [3] requires a much stronger external field $E_0$ than is needed to excite the fundamental mode. In the experiments [4-10] jet formation is observed with a certain time lag at the field strength corresponding to the appearance of instability of the fundamental mode. Accordingly, in order to explain the experimental facts, Taylor [4] proposed a phenomenological model which reduces to the jets being drawn out by the external field from the tips of the conical projections on the surface of the liquid. The mechanism of formation of these conical projections was not considered in [4]. By examining the excitation of instability of the higher capillary wave modes on a surface with a curvature that varies with time, we will show that Taylor's objections to the Rayleigh model are groundless: the higher modes can be excited in the same external field as the fundamental mode if the dependence of the electric field strength at the surface of the drop on its curvature is taken into account.

1. Let a stationary, perfectly conducting drop of incompressible inviscid fluid of radius $R$ in a nonconducting ideal incompressible medium with permittivity equal to unity be located in a homogeneous electric field $E_0$. The equilibrium shape of the drop in the external field is close to that of an ellipsoid revolution prolate along the field [11-13]. We assume the shape of the drop to be ellipsoidal ($e$ is the eccentricity) and in the $e^2 \ll 1$ approximation, using a spherical coordinate system moving with the center of the drop, we solve the problem of the stability in the external field $E_0$ of the capillary waves already existing in the drop owing to the thermal motion of the molecules [14].

The equation of an ellipsoid prolate along the field (the polar angle is reckoned from $E_0$) has the form:

$$\eta(\theta) = \frac{r(\theta)}{a} = \frac{1-e^2}{\sqrt{1-e^2 \cos^2 \theta}}$$
where $a$ and $e^2$ are the semimajor axis of the ellipsoid and the square of its eccentricity.

In the drop and the surrounding medium let there exist a motion of infinitely small amplitude with velocity potentials $\psi_i(r, \theta, \varphi, t)$ ($i = 1$ inside, and $i = 2$ outside the drop) leading to the distortion of its shape $r(\theta, \varphi, t) = r(\theta) + \xi(\theta, \varphi, t); \ |\xi| \ll R$. The potentials $\psi_1$ and, moreover, the potential $\Phi$ of the electric field outside the drop are harmonic functions satisfying the following boundary conditions at the surface of the drop [12, 15]:

\begin{equation}
\Delta \psi_i = 0; \quad i = 1, 2; \quad \Delta \Phi = 0
\end{equation}

\begin{equation}
\frac{\partial \psi_i}{\partial n_i} = -\frac{\partial \psi_2}{\partial n_2} = \frac{\partial \psi_1}{\partial n_1}; \quad \frac{\partial \xi}{\partial t} \approx \frac{\partial \psi}{\partial n}; \quad (r = r(\theta))
\end{equation}

\begin{equation}
\Phi(r(\theta) + \xi(\theta, \varphi, t)) = 0
\end{equation}

\begin{equation}
\Delta p = -\rho_1 \frac{\partial \psi_1}{\partial t} + \rho_2 \frac{\partial \psi_1}{\partial t} + f(\theta) = \sigma \left( \frac{1}{R^2} + \frac{1}{R''^2} \right)
\end{equation}

where $f$ is the total pressure exerted on the interface by the external and perturbed electric fields, $\sigma$ is the surface tension, $R'$ and $R''$ are the principal radii of curvature of the surface at the point in question. All the derivatives in (1.2)-(1.4) are referred to the undisturbed surface of the ellipsoid, and in the approximation $e^2 << 1$ to the surface of a sphere of radius $R$. In the same approximation $e^2 << 1$ in boundary conditions (1.2) the derivatives along the normal to the surface $\partial \psi/\partial n$ can be replaced by $\partial \psi/\partial r$, the derivatives with respect to the radial coordinate.

In spherical coordinates when $e^2 << 1$ in the approximation linear in $\xi/R$ the Laplace pressure under the ellipsoidal surface distorted by the wave motion has the form

\begin{equation}
\sigma \left( \frac{1}{R^2} + \frac{1}{R''^2} \right) = -\sigma \left( \frac{2}{R^2} - \frac{1}{R' R''} L^+ \right) \xi - \frac{\sigma}{R^2} e^2 [3(1 + e^2) \sin^2 \theta - 2] \xi
\end{equation}

Here $L^+$ is the Legendre operator (the angular part of the Laplacian operator in spherical coordinates taken with a minus sign). The first term in (1.5) is the pressure due to the distorted spherical surface [15], while the second term represents the component associated with the ellipsoidal shape of the drop.

2. For the velocity potentials $\psi_i$, the electric field potential $\Phi$, and the perturbation $\xi$ we assume: $\psi_1, \psi_2, \Phi, \xi \sim \exp(i\omega t)$. Substituting this time dependence in (1.4) with account for (1.5) and differentiating the relation obtained once with respect to $t$ for constant $\theta$ and $\varphi$, in the approximation linear in $\xi/R$ taking into account the boundary conditions (1.2) we obtain

\begin{equation}
(\rho_1 \psi_1 - \rho_2 \psi_2) \omega^2 + \frac{\partial f}{\partial t} + \frac{\sigma}{R^2} [2 - L^+] \frac{\partial \psi_2}{\partial r} + \frac{2\sigma e^2}{R^2} [3(1 + e^2) \sin^2 \theta - 2] \frac{\partial \psi_1}{\partial r} = 0 \quad (r = R)
\end{equation}

The expression for the partial derivative with respect to time of the electrostatic field pressure on the surface of a slightly ellipsoidal drop is written, in the approximation linear in $\xi/R$, in the form:

\begin{equation}
\frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{E^2}{8\pi} \right]_{r=r(\theta) + \xi/8\pi} \approx \frac{\partial}{\partial t} \left[ \frac{1}{8\pi} \frac{\partial E^2}{\partial r} \xi + \frac{1}{4\pi} E \cdot \delta \xi \right]_{r=r(\theta)}
\end{equation}

Introducing the notation $\Phi = \phi + \delta \Phi$, $E = -\nabla \Phi$, $\delta E = -\nabla (\delta \Phi)$, from (1.1)-(1.3) we obtain

\begin{equation}
r = r(\theta); \quad \delta \Phi \approx \frac{\partial \Phi_0}{\partial r} \approx -\xi E_0
\end{equation}

where $\Phi_0$ and $E_0$ are the potential and strength of the electrostatic field on the undisturbed ellipsoidal surface. Using (2.3), we reduce (2.2) to the form:

\begin{equation}
\frac{\partial f}{\partial t} = \frac{E_0^2}{4\pi} \frac{\partial^2 \xi}{\partial r \partial t} \Bigg|_{r=r(\theta)} \approx \frac{E^2}{4\pi} \frac{\partial^2 \psi}{\partial r^2} \Bigg|_{r=R}
\end{equation}