A modification to the model developed in [1] for the nonequilibrium filtration of an inhomogeneous liquid is proposed. The structure of the saturation discontinuity is investigated, taking into account the nonequilibrium conditions; it is shown to be stable in a linear approximation.

1. Basic Equations

The displacement of one liquid by another with which it is immiscible is described by the system of equations [2]

\[
\begin{align*}
    m \frac{\partial \sigma}{\partial t} + \text{div} V_1 &= 0, \\
    m \frac{\partial (1-\sigma)}{\partial t} + \text{div} V_2 &= 0 \\
    V_i &= -\frac{k_i}{\mu_i} \text{grad} p_i, \quad (i=1, 2), \\
    p_1 - p_2 &= p_c
\end{align*}
\]

(1.1)

where \( m \) is the porosity at the medium; \( \sigma \) is the saturation of the pore space by the displacing phase (denoted by the subscript 1; the phase being displaced is denoted by the subscript 2); \( V_1 \) and \( V_2 \) are the rates of filtration of the phases; \( k \) is the absolute permeability of the porous medium; \( \mu_1 \) and \( \mu_2 \) are the viscosities of the phases; \( P_1 \) and \( P_2 \) are the pressures of the phases; \( f_1 \) and \( f_2 \) are the relative permeabilities of the phases; \( p_c \) is the static capillary pressure.

The relative phase permeability is determined experimentally under conditions of steady filtration and in classical filtration theory it is assumed that the function determined in this way, which depends on the single value of the saturation \( \sigma \) (Fig. 1), also characterizes the resistance to motion of the phases under nonsteady conditions. Analogously, it is assumed in classical filtration theory that the capillary pressure \( p_c(\sigma) \) is equal to the equilibrium value even under nonsteady conditions,

\[
p_c = T \cos \theta \frac{\mu_i}{\mu} \phi(\sigma)
\]

(1.2)

where \( T \) is the surface tension between the phases; \( \theta \) is the wetting angle; \( \phi(\sigma) \) is the Leverett function [3].

The case when the displacing liquid is more wetting than the liquid being displaced is shown in Fig. 1.

In [1] it was suggested that nonequilibrium effects may be taken into account by choosing values on the equilibrium phase-permeability and capillary-pressure curves not at the point corresponding to the instantaneous value of the saturation \( \sigma(t, x) \), but at the point \( \sigma(t, \pm \tau, x) \). A mechanism was proposed for the internalization of part of the liquid in the course of the adjustments in the filtration process as the saturation of the displacing liquid increases.

According to this model, the displacement may be characterized by the relations

\[
\begin{align*}
    f_i &= f_i(\sigma(t-\tau, x)) \approx f_i(\sigma - \tau \partial_x \sigma), \\
    f_i &= f_i(\sigma(t+\tau, x)) \approx f_i(\sigma + \tau \partial_x \sigma) \\
    p_c &= p_c(\sigma(t-\tau, x)) \approx p_c(\sigma - \tau \partial_x \sigma)
\end{align*}
\]

(1.3)

where \( \tau \) is the displacement time, the characteristic time for the relaxation of the adjustments in the saturation field.

Thus, the values of \( f_i \) and \( f_2 \) are less here than in the classical theory and \( p_c \) is larger.

Although the displacing liquid may be internalized, which in fact reduces its phase permeability, the internalized part of the displaced liquid in the flow of displacing liquid acts as a well — it is completely closed by capillary seals and may be simply replaced by the displacing liquid. Therefore, the effective saturation of
the displacing liquid must be larger than the real saturation. Thus, the internalization mechanism leads, on
the one hand, to decrease in \( f_\text{i} \) and, on the other, to increase in \( f_\text{f} \). Since the internalized part of the displaced
liquid is evidently much larger than the internalized part of the displacing liquid, the second of these processes
is dominant.

The capillary pressure is partly dissipated in retaining the internalized displaced liquid by means of
capillary seals, and therefore this part has no effect on the overall motion. Hence, \( p_\text{C} \) must be taken to be less
than in classical theory.

The model proposed in the present paper is based on the assumption that the relative permeabilities of
the two phases and the capillary pressure correspond to the equilibrium values at some higher value of the satu-
ration, so that

\[
\begin{align*}
\alpha_\text{i} &= f_\text{i}(\alpha(t+\tau, x)) \approx f_\text{i}(\alpha + \Delta \alpha), \\
\alpha_\text{f} &= f_\text{f}(\alpha(t+\tau, x)) \approx f_\text{f}(\alpha + \Delta \alpha), \\
p_\text{f} &= p_\text{f}(\alpha(t+\tau, x)) \approx p_\text{f}(\alpha + \Delta \alpha)
\end{align*}
\]

The dependence of the displacement time \( \tau \) on the saturation is neglected, and \( \tau \) is assumed constant.

Passing to dimensionless parameters [1], the following system of equations is obtained:

\[
\begin{align*}
m&\partial_\text{i} \alpha + \text{div} \text{ V}_1 = 0, \quad V_1 = -f_\text{i}(\sigma + \varepsilon_1 \partial_\text{i} \sigma) \text{grad}_1 P_1 \\
-m&\partial_\text{f} \sigma + \text{div} \text{ V}_2 = 0, \quad V_2 = -f_\text{f}(\sigma + \varepsilon_2 \partial_\text{f} \sigma) \text{grad}_2 P_2 \\
\mu_1 &- \mu_2 < 1, \quad \varepsilon_1 = \frac{\tau}{\tau_1}, \quad \varepsilon_2 = \frac{T \cos \theta}{\Delta P} \sqrt{\frac{m}{k}}, \quad \tau = \frac{\mu_2 L^2}{k \Delta P}
\end{align*}
\]

where \( \tau \) is the characteristic time; \( L \) is the characteristic dimension of the region in which filtration occurs;
\( \Delta P \) is the characteristic pressure difference (from the boundary conditions).

For most problems, \( \varepsilon_1 \) and \( \varepsilon_2 \) are small. To obtain a first approximation to the solution, \( \varepsilon_1 \) and \( \varepsilon_2 \) are
neglected (this is permissible if \( \partial_\text{f} \sigma \) and \( \text{grad}_2 \sigma \) are small), leading to the classical Muskat–Mires system
of equations for the first term of the external expansion [1, 4]. Generally speaking, this system leads to dis-
continuous solutions. A saturation discontinuity appears, and the calculation of its motion requires the imposi-
tion of conservation laws. At discontinuities \( \partial_\text{f} \sigma \) and \( \text{grad}_2 \sigma \) are infinite, i.e., the basic conditions of the ex-
ternal expansion are not satisfied. Therefore, it is necessary to consider the internal expansion, without
neglecting \( \varepsilon_1 \) and \( \varepsilon_2 \), and since this approach is only necessary in a small vicinity of the discontinuity front,
the motion may be regarded as one-dimensional.

In the vicinity of the discontinuity surface, the following substitutions are made: \( \theta = t\delta; \xi_1 = x_1; \xi_2 = x_2; \xi_3 = x_3 \)
(assuming that the discontinuity surface is perpendicular to the axis \( \xi_1 \)), where \( \delta \) is a small parameter. In
addition, it is clear that since \( \varepsilon_1 \) and \( \varepsilon_2 \) are small, the term \( \varepsilon_2 \partial_\text{f} \sigma \) in the expression \( \varepsilon_1 J(\sigma + \varepsilon_2 \partial_\text{f} \sigma) \) gives an effect
of higher order, which may be neglected in the first approximation, therefore, this expression may be taken in
the form \( \varepsilon_1 J(\sigma) \).

Neglecting small quantities, the result is

\[
\begin{align*}
m&\partial_\text{i} \sigma + \partial_\text{i} V_1 = 0, \quad V_1 = -f_\text{i}(\sigma + \delta_1 \partial_\text{i} \sigma) \frac{1}{\delta_1} \partial_\text{i} P_1 \\
-m&\partial_\text{f} \sigma + \partial_\text{f} V_2 = 0, \quad V_2 = -f_\text{f}(\sigma - \delta_1 \partial_\text{f} \sigma) \frac{1}{\delta_2} \partial_\text{f} P_2, \quad P_2 = P_1 = \varepsilon_1 J(\sigma)
\end{align*}
\]