HYDRODYNAMIC DRAG OF AN ELLIPSOIDAL DROPLET AT SMALL REYNOLDS NUMBERS

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At present, there is an absence of the accurate data on the influence of the shape of a droplet on its hydrodynamic drag and mass transfer without which the design of mass transfer apparatus is impossible [1-3]. Most often it is assumed that the drag of an ellipsoidal liquid droplet as it moves along the axis of symmetry is determined by the product of the drag of a spherical liquid droplet and a coefficient which takes into account the shape and is determined from the drag of a solid ellipsoid for which the exact solutions are known. It is shown below that this assumption is not always valid.

A droplet of polarizable liquid in an electric field and a droplet of magnetic liquid in a magnetic field are pulled out along the field, acquiring a shape near ellipsoidal [3, 4]. At the same time, the elongation of the droplets is determined by the balance of electromagnetic and surface tension forces and is proportional to the parameter \( S = \mu_0 M^2 r_0 / \alpha \) for a magnetic liquid [3] (\( \mu_0 \) is the permeability of the vacuum, \( M \) is the magnetization of the liquid, \( \alpha \) is the surface tension) and to the parameter \( \varepsilon_0 E^2 r_0 / \alpha \) in the electric field [4] (\( \varepsilon_0 \) is the permittivity of the vacuum, \( E \) is the electric field strength). In the case of electrodynamics elongation transition from a sphere to both a prolate and an oblate ellipsoid is possible, depending on the ratio of the conductivities of the liquids inside and outside the droplet. We confine ourselves below to a consideration of the motion of a prolate ellipsoid along its axis of symmetry. An actual droplet can be assumed an ellipsoid right up to the semiaxis ratio \( a/b \approx 5 \) [5]. From the point of view of hydrodynamics, the motion of an actual droplet does not differ greatly from the motion of an ellipsoidal droplet even in the case of large elongations.

A liquid ellipsoid can be assumed undeformable if the stresses exerted by the exterior flow with order \( \eta U/r_0 \) (\( \eta \) is the viscosity of the external liquid, \( U \) is the velocity of the droplet, \( r_0 \) is the radius of the spherical droplet) are much less than the capillary forces which determine its shape. These last have order \( 2 \gamma / R_1 \), where \( R_1 \) is the radius of curvature at the vertex of the prolate ellipsoid. Since \( R_1 = r_0 (b/a)^{1/3} \), the droplet can be assumed undeformable when the condition \( A \ll 1 \) is satisfied:

\[
A = \frac{U \eta}{2 \alpha} \left( \frac{b}{a} \right)^{1/3} \ll \frac{U \eta}{\alpha}
\]  

(1)

For droplets falling in a liquid of viscosity \( \eta = 10^{-1} \) kg·m⁻¹·sec⁻¹ with a coefficient of interphase tension \( \alpha = 10^{-2} \) N/m, the condition \( A \ll 1 \) is satisfied for \( U \ll 10 \) cm/sec, and this corresponds to the requirement for small Reynolds numbers.

Thus, this paper considers the motion of a prolate ellipsoidal undeformable droplet along the axis of symmetry of the ellipsoid at small Reynolds numbers under the influence of a certain force (gravitational, magnetic, or electrophoretic).

We introduce the system of coordinates of a prolate ellipsoid of revolution \( \sigma, \tau, \varphi \) [6] so that the coordinate surface \( \sigma = \) const is the same as the surface of the droplet. For an axisymmetric flow which does not depend on the angle \( \varphi \), the continuum equation \( \nabla \cdot \mathbf{v} = 0 \) makes it possible to introduce the stream function \( \psi \) connected to the velocity components by the relations

\[
\nu_\sigma = -\frac{1}{c^2 \gamma (\sigma^2 - \tau^2) (\sigma^2 - 1)} \frac{\partial \psi}{\partial \tau}; \quad c = \frac{r_0}{1 - \sigma^2 (\sigma^2 - 1)}; \quad \nu_\tau = -\frac{1}{c^2 \gamma (\sigma^2 - \tau^2) (1 - \tau^2)} \frac{\partial \psi}{\partial \sigma}
\]

(2)

where \( \nu_\sigma \) and \( \nu_\tau \) are the velocity vector components \( (\nu_\varphi = 0) \), \( 2c \) is the distance between the
foci of the ellipsoid (the equation for c is obtained from the condition of constancy of the volume of the ellipsoid).

The Stokes equations which describe the steady motion of a droplet at small Reynolds numbers, expressed in dimensionless form for the variables stream function-vorticity, take the form \[7\]

\[
\omega = \frac{1}{\sqrt{(\sigma-1)(1-\tau)}} E^2 \psi, \quad E^2 [Y(\sigma-1) (1-\tau)] \omega = 0, \quad E^2 = \frac{1}{\sigma^2-1} \left[ (\sigma^2-1) \frac{\partial^2}{\partial \sigma^2} + (1-\tau) \frac{\partial^2}{\partial \tau^2} \right]
\]

where the vorticity \(\omega\) is defined as \(\omega = \text{curl} \, v\), while the velocity of the droplet \(U\) and half the distance between the foci of the ellipsoid \(c\) are selected as the scales. In this case, the semi-axes of the ellipsoid are \(a = c_0, b = c \sqrt{c_0(c_0-1)}\).

The boundary conditions in the system of coordinates associated with the droplet have the form

\[
\psi_s = \frac{1}{2} (\sigma^2-1)(1-\tau^2), \quad \sigma = \sigma_0; \quad \psi_s = 0, \quad \omega = 0, \quad \sigma = 1
\]

\[
\psi_s = 0, \quad \psi_s = 0, \quad \psi_s = 0, \quad \omega = 0, \quad \omega = 0, \quad \tau = \pm 1
\]

Here, as below, the subscripts \(i\) and \(e\), respectively, identify the quantities which characterize the liquid inside and outside the droplet.

The conditions of equality of the velocities and the following equation of tangential stresses must be satisfied at the interface \(\sigma = \sigma_0\):

\[
\psi_s = 0, \quad \psi_s = 0, \quad \frac{\partial \psi_s}{\partial \sigma} = \frac{\partial \psi_s}{\partial \sigma}, \quad \mu \frac{\partial}{\partial \sigma} \left[ \frac{1}{(\sigma^2-1) \frac{\partial \psi_s}{\partial \sigma}} - \frac{1}{(\sigma^2-1) \frac{\partial \psi_s}{\partial \sigma}} \right], \quad \mu = \eta \eta
\]

The force exerted by the flow on the droplet is

\[
F_s = \int \Pi_{ee} dS = \int \{ \Pi_{ee} e_i e_i + \Pi_{ee} e_i e_i \} dS
\]

where the tensors \(\Pi_{sq}, \Pi_{st}\) are the tensor components of the viscous stresses in elliptic coordinates, \(S\) is the surface of the ellipsoid, \(e_i\) and \(e_r\) are unit vectors in the coordinate system of the prolate ellipsoid, \(e_0\) is the unit vector directed along the axis of symmetry. With allowance for the explicit form of \(\Pi_{sq}\) and \(\Pi_{st}\) and the fact that \(e_0 e_0 = \eta/((\sigma^2-1)(\sigma^2-1))\), expression (6) takes the following form after simple transformations:

\[
F_s = \pi \eta \eta e U \left[ \frac{(\sigma^2-1)^2}{\sigma_0 c_0 (\sigma^2-1)} \right] \int_{-1}^{1} \left[ \frac{1-(\sigma^2-1)^2}{\sigma^2-1} \omega_s \right] d\tau
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\]

Since the influence of the shape of the droplet on its drag is of interest, it is natural to select the drag force of the spherical droplet, \(F_s = 2 \pi \eta e U (3 \mu + 2)/(\mu + 1)\), as the scale factor [8]. The aim of the paper is to determine the shape factor \(k_F\) determined by the relation \(k_F = F_s/F_s^f\) and characterizing the influence of the shape of the liquid droplet on its hydrodynamic drag. This factor is determined by the expression

\[
k_f = \eta e U \left[ \frac{(\sigma^2-1)^2}{\sigma_0 c_0 (\sigma^2-1)} \right] \int_{-1}^{1} \left[ \frac{1-(\sigma^2-1)^2}{\sigma^2-1} \omega_s \right] d\tau
\]

and to determine it, problem (3) with boundary conditions (4), (5) must be solved and \(\omega_s\) found.

Condition (5) means that the separation of variables is impossible and, therefore, hinders the analytic solution of the problem. Therefore, the formulated problem was solved numerically by the grid method. The computational region \(-1 \leq \tau \leq 1, 1 \leq \sigma \leq \sigma_0\) corresponding to the meridian section of the ellipsoid was covered by a uniform grid with the step \(h_{\tau} = 0.1\) with respect to \(\tau\) and with the step \(h_{\sigma} = (\sigma_0 - 1)/20\) with respect to \(\sigma\) in the interior region. In the exterior region, the grid was selected as nonuniform with respect to \(\sigma\), with a step increasing in accordance with the linear law with increasing distance from the surface of the ellipsoid. The second differential derivatives