MODELING PRESSURE-STRAIN RATE CORRELATIONS IN TURBULENCE THEORY

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On the basis of a spectral representation of the "rapid" part \( \varphi_{ij,2} \) of the correlation tensor \( \langle \rho(\partial u_i/\partial x_j) \rangle \) using Cramer's theorem the inequality \( \varphi_{ij,2}(\partial U_i/\partial x_j) \geq 0 \) is obtained. As distinct from the realizability conditions, it can serve as a direct and very rigorous test of the adequacy of model expressions for \( \varphi_{ij,2} \). In particular, it is shown that the best known of such expressions do not satisfy this test.

The first term on the right side of (1.1) is assumed to play a dominant role; therefore, instead of (1.1), the so-called quasi-isotropic model is often employed:

\[
(\varphi_{ij,2})_s = -\gamma (P_{ij}^{-1/2} \delta_{ij})
\]

Relation (1.2) admits a simple physical interpretation: the correlations \( \langle \rho(\partial u_i/\partial x_j) \rangle \) equalize the degrees of generation of the various components of \( \langle u_i u_j \rangle \).

A comparison with experiment makes it possible to propose the approximate values 0.6 and 0.4 for the constants \( \gamma \) and \( C_2 \), respectively.

The above-mentioned difficulty associated with representations (1.1), (1.2) and the like consists in the fact that when any of them are used the satisfaction of the so-called realizability conditions for solutions of the system of transport equations is not guaranteed. These conditions include, in particular, the inequalities

\[
\langle u_i^2 \rangle \geq 0; \quad \langle u_j^2 \rangle \geq 0; \quad \langle u_i u_j \rangle \leq \langle u_i^2 \rangle \langle u_j^2 \rangle
\]

As noted in [2], the probability of these conditions being unintentionally infringed is considerable. In order to exclude this possibility, it is necessary to reject the assumption that the coefficients (for example, \( C_2 \)) are constant and model their dependence on the invariants of the tensor \( \langle u_i u_j \rangle \). Such attempts were made, for example, in [2, 4, 5], and very cumbersome
expressions, which have not yet been used in practical calculations, were obtained for \( \varphi_{ij,2} \). Moreover, despite the use of ideas with a rich physical content, these expressions were based on formal interpolation formulas, so that the problem of realizability (and especially the satisfaction of the last of conditions (1.3)) in fact remains.

In this study we derive an inequality that sets a limit on the possible values of the components of the tensor \( \langle p(\partial u_i/\partial x_j) \rangle \). As distinct from the conditions of realizability, this inequality can serve as a direct criterion of the adequacy of the models in the sense that it can be tested without solving the system of transport equations.

2. Using the incompressibility condition, for the part of the fluctuating pressure \( p \) determined by the nonuniformity of the mean velocity we easily obtain the relation

\[
p(x) = \frac{1}{2\pi} \left( \frac{\partial u_i(x')}{\partial x_j} \right) \frac{dx'}{|x-x'|}
\]

(2.1)

From (2.1), taking into account the condition of local uniformity \( l \ll L \) (\( l \) is the integral scale of turbulence, and \( L \) is the characteristic external length scale), we have [6]

\[
\langle p(x_i) \left( \frac{\partial u_i(x_j)}{\partial x_j} \right) \rangle = \frac{1}{2\pi} U_{1m}(x) \int \frac{\partial^2 R_{im}(x, r)}{\partial r_i \partial r_j} \frac{dx'}{|x-x'|}
\]

(2.2)

\[R_{im}(x, r) = \langle u_i(x) u_m(x+r) \rangle, \quad x = \frac{1}{2}(x_i+x_j), \quad r=x_i-x_j\]

Here, \( R_{im} \) is the velocity correlation tensor.

From (2.2), using the spectrum tensor \( F_{ij}(x, k) \), we obtain the representation for \( \varphi_{ij,2} \) [2, 4]

\[
\varphi_{0,2} = \frac{8U_{1m}}{(2\pi)^2} \int R_{im}(x, r) \exp(-ikr)dr, \quad \theta = \frac{k}{k}
\]

(2.3)

Relation (2.3) is the starting point for the semiempirical modeling of the quantity \( \varphi_{ij,2} \), the main difficulty being finding the components of the fourth-order tensor \( f_{ilm}^{(ij)} \). They must satisfy the following obvious conditions:

\[
f_{im}^{(ij)} = f_{m}^{(ij)} = f_{1}^{(ij)}, \quad f_{im} = \langle u_i u_m \rangle, \quad f_{1m}^{(ij)} = 0
\]

However, these conditions are insufficient for determining all 81 components of the tensor and further assumptions must be made. Thus, in [4] it is assumed that the eigenvectors of the tensors \( f_{ilm}^{(ij)} \) and \( \langle u_i u_j \rangle \) coincide; as a result, the number of unknown components can be reduced to three.

In [2] it is assumed that the tensor \( f_{ilm}^{(ij)} \) is an isotropic function of the anisotropy tensor \( b_{ij} = \langle u_i u_j \rangle / \langle u_2^2 \rangle - (1/3)\delta_{ij} \), which gives nine unknown coefficients that depend on the invariants of the tensor \( b_{ij} \). If, moreover, we confine ourselves to the approximation linear in \( b_{ij} \), only one coefficient, which is usually assumed to be constant, remains unknown. As a result, for \( \varphi_{ij,2} \) we obtain the expression

\[
\varphi_{ij,2} = -\frac{2}{3} U_{ij} \langle u_i u_j \rangle + \frac{2}{3} U_{i} \langle u_i u_j \rangle + \frac{1}{15} \langle u_i^2 \rangle (4U_{ij} - U_{ij})
\]

\[+ 2\delta_{ij} U_{im} \langle u_i u_m \rangle + \frac{4}{3} U_{ij} \langle u_i u_j \rangle + \frac{4}{3} U_{ij} \langle u_i u_j \rangle - \frac{4}{3} U_{ij} \langle u_i u_j \rangle
\]

(2.4)

\[C = \text{const.}\]

Symmetrizing (2.4) with respect to the indices \( i, j \) we obtain the representation (1.1) with \( C_2 = -2/3(11C + 1) \).

In [7] the modeling is based on the representation of the spectrum tensor \( F_{ij} \) in the form of an isotropic function of the vector \( \theta \) and the tensor \( b_{ij} \).

However, from (2.3) we can directly derive important limitations on the components of the tensor \( \varphi_{ij,2} \) without making any additional assumptions. For this purpose we use Cramer’s theorem [8], according to which

\[
F_{ij} \xi^{i} \xi^{j} = 0
\]

(2.5)

Here, \( \xi \) is an arbitrary vector, and a bar denotes the complex conjugate.

If the vector \( \xi \) does not depend on \( \theta \), after integration over all \( k \) condition (2.5) reduces to the well-known limitation: the matrix \( \langle u_i u_j \rangle \) must be positive definite and, in particular, conditions (1.3) must be satisfied.

On the other hand, choosing as \( \xi \) the vector \( U_{ij} \theta_i \), after integrating (2.5) over all \( k \) we obtain