The Abstract Variable-binding Calculus

Abstract. The abstract variable binding calculus (VB-calculus) provides a formal framework encompassing such diverse variable-binding phenomena as lambda abstraction, Riemann integration, existential and universal quantification (in both classical and nonclassical logic), and various notions of generalized quantification that have been studied in abstract model theory. All axioms of the VB-calculus are in the form of equations, but like the lambda calculus it is not a true equational theory since substitution of terms for variables is restricted. A similar problem with the standard formalism of the first-order predicate logic led to the development of the theory of cylindric and polyadic Boolean algebras. We take the same course here and introduce the variety of polyadic VB-algebras as a pure equational form of the VB-calculus. In one of the main results of the paper we show that every locally finite polyadic VB-algebra of infinite dimension is isomorphic to a functional polyadic VB-algebra that is obtained from a model of the VB-calculus by a natural coordinatization process. This theorem is a generalization of the functional representation theorem for polyadic Boolean algebras given by P. Halmos. As an application of this theorem we present a strong completeness theorem for the VB-calculus. More precisely, we prove that, for every VB-theory $T$ that is obtained by adjoining new equations to the axioms of the VB-calculus, there exists a model $D$ such that $\vdash_T s = t$ iff $\models_D s = t$. This result specializes to a completeness theorem for a number of familiar systems that can be formalized as VB-calculi. For example, the lambda calculus, the classical first-order predicate calculus, the theory of the generalized quantifier exists uncountably many and a fragment of Riemann integration.

1. Introduction

This paper is the outgrowth of an attempt to develop a general algebraic theory of variable-binding and variable-substitution that encompasses both functional abstraction, as formalized in the lambda calculus, and quantification. It was our hope that it might even comprehend variable-binding phenomena that on their face seem far removed from either functional abstraction or quantification, for instance, axiomatic formalizations of fragments of integration theory.

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Functional abstraction and quantification have many properties in common. This is easiest to see at the formal level; for example invariance under change of bound variable is a characteristic property of both. (In the standard formalization of the lambda calculus this property appears explicitly as the axiom of $\alpha$-conversion.) It is less obvious and perhaps surprising that this compatibility is evident also at the semantical level. It turns out that these similarities can be exploited to give a general theory of variable-binding (and variable-substitution) that is surprisingly rich in structure.

The variable-binding calculus (VB-calculus or simply VBC for short) is modeled on the lambda calculus; in particular all of its axioms are in the form of equations. Roughly speaking, it is obtained from the lambda calculus by omitting the axioms that deal with the interaction between functional abstraction and application and retaining, apart from the axioms and inference rules of equality, only $\alpha$-conversion. A VB-theory is obtained by adjoining new equations as additional axioms. Thus the lambda calculus itself can be thought of as a VB-theory. Most quantifier theories can also be interpreted in a natural way as VB-theories. This applies in particular to the classical first-order predicate calculus $L$ and to the theory $L(Q)$ of the generalized quantifier exists uncountably many. In particular these traditionally assertional logical systems are transformed in this context into equational calculi. We shall show how even Riemann integration of polynomials with multiple indeterminates can be interpreted as a VB-theory.

In one of the two main results of the paper, Thm. 3.10, we obtain a strong completeness theorem for the VB-calculus that specializes to give a completeness theorem for every VB-theory. In particular it specializes to a completeness theorem for the lambda calculus that is very close in spirit to the ones formulated in Meyer [20] in terms of the so-called environment models and in Barendregt [4] in terms of syntactical models. It also gives completeness theorems for the equational calculi versions of $L$ and $L(Q)$, but the connection between them and the well known completeness theorems for the assertional versions of these two logics is a more subtle question.

When one thinks about how the assertional formalization $L$ of classical first-order logic is traditionally algebraized, and then compares this with the lambda calculus, it becomes clear that the algebraization of an assertional logic can be viewed as a two-step process. The conversion of $L$ into an equational calculus on the model of the lambda calculus constitutes the first step. But, although the lambda calculus is a calculus of equations, it is not an equational theory in the pure sense because unrestricted substitution for variables in a valid equation does not in general result in another valid equation due to possible clash of variables. In the theory of the lambda cal-