Abstract. In the paper [8], the first author developed a topos-theoretic approach to reference and modality. (See also [5]). This approach leads naturally to modal operators on locales (or “spaces without points”). The aim of this paper is to develop the theory of such modal operators in the context of the theory of locales, to axiomatize the propositional modal logics arising in this context and to study completeness and decidability of the resulting systems.

1. Modal operators on locales

We shall follow both the terminology and the notation of [4] and let \( \text{Loc}(S) \) be the category of locales of a topos \( S \) which the reader may think of the category of sets. Recall that in the particular case in which \( S \) is in fact the category of sets, the objects of \( \text{Loc}(S) \) are posets admitting arbitrary suprema and (hence) arbitrary infima for which the distributive law

\[
\bigvee_{i \in I} b \wedge a_i = b \wedge \bigvee_{i \in I} a_i
\]

holds. A morphism is a function (between the underlying sets) which preserves arbitrary suprema and finite infima.

The category of spaces of \( S \), \( \text{Sp}(S) \), is defined to be the opposite of the category of locales of \( S \), ie, \( \text{Sp}(S) = \text{Loc}(S)^o \). In other words, both categories have the same objects but reverse morphisms. It is convenient, however to use different notations for objects in these categories. If \( X \) is an object of \( \text{Sp}(S) \), we let \( O(X) \) the same object in \( \text{Loc}(S) \). Similarly, if \( f \) is a morphism of \( \text{Sp}(S) \), then \( f^- \) will be the corresponding morphism of \( \text{Loc}(S) \). Thus, by the very definition of these categories, we have the equivalence

\[
f : X \rightarrow Y \in \text{Sp}(S) \\
f^- : O(Y) \rightarrow O(X) \in \text{Loc}(S)
\]

Finally, we recall that the morphism \( f^- \) has always a right adjoint \( f_* \), ie, \( f^- \dashv f_* \).

**Definition 1.1.** Let \( q : L \rightarrow Q \) be a morphism in \( \text{Sp}(S) \).
1. \( q \) is a quotient if \( q \cdot q' = id_{\mathcal{O}(Q)} \)

2. \( q \) is an essential quotient if it is a quotient and \( f' \) has a left adjoint \( \exists f \).

3. \( q \) is an open quotient if it is an essential quotient and for \( a \in \mathcal{O}(L) \) and \( b \in \mathcal{O}(Q) \)
   \[ \exists q(q'(b) \wedge a) = b \wedge \exists q(a) \]

4. \( q \) is a Boolean quotient if it is an open quotient and \( \mathcal{O}(Q) \) is a Boolean algebra

5. \( q \) is a strong Boolean quotient if it is a Boolean quotient and \( \mathcal{O}(Q) \) is the Boolean algebra of the complemented objects of \( \mathcal{O}(L) \).

**Definition 1.2.** Let \( L \in Sp(S) \) and let \( \Box, \Diamond : \mathcal{O}(L) \longrightarrow \mathcal{O}(L) \) be a pair of operators in \( S \).

1. \( \Box \) is a necessity operator on \( L \) iff
   
   (a) \( a \leq b \implies \Box a \leq \Box b \)
   
   (b) \( \Box \leq Id_{\mathcal{O}(L)} \)
   
   (c) \( \Box T = T \) where \( T \) is the largest element of \( \mathcal{O}(L) \)
   
   (d) \( \Box(a \wedge b) = \Box(a) \wedge \Box(b) \)
   
   (e) \( \Box^2 = \Box \)

2. The pair \( (\Diamond, \Box) \) is a MAO_0 couple if

   (a) \( \Diamond \vdash \Box \)

   (b) \( \Box \leq id_{\mathcal{O}(L)} \leq \Diamond \)

   (c) \( \Box^2 = \Box, \Diamond^2 = \Diamond \)

3. The pair \( (\Diamond, \Box) \) is a MAO couple if it is a MAO_0 couple and satisfies the Frobenius condition
   \[ \Diamond(a \wedge \Box b) = \Diamond a \wedge \Box b \]

4. The pair \( (\Diamond, \Box) \) is an IBM couple if it is a MAO couple and, furthermore, satisfies excluded middle for necessary formulas
   \[ \Box a \lor \neg \Box a = T \]

5. The pair \( (\Diamond, \Box) \) is an \( IBM^c \) couple if it is an IBM couple and satisfies the condition
   \[ a \lor \neg a = T \iff a = \Box a \]