CLASSIFICATION OF SIMPLE $\mathfrak{sl}(2)$-MODULES AND FINITE-DIMENSIONALITY
OF THE MODULE OF EXTENSIONS OF SIMPLE $\mathfrak{sl}(2)$-MODULES

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A classification of simple $\mathfrak{sl}(2)$-modules is carried out, and the finite-dimensionality of $\text{Ext}_{\mathfrak{sl}(2)}(M)$, where $M$ and $N$ are simple $\mathfrak{sl}(2)$-modules is established.

In this paper we give the classification of simple $\mathfrak{sl}(2)$-modules up to prime elements of a certain Euclidean ring, and we prove the finite-dimensionality of the module of extensions of simple $\mathfrak{sl}(2)$-modules. We show that any operator from the universal enveloping algebra, which acts nontrivially (i.e., it is different from the zero operator) on a simple module, has finite-dimensional kernel and cokernel. We compute the module of extensions for a sufficiently wide class of simple modules. We consider a certain series of simple modules which generalizes in a natural manner the Whittaker modules and the Arnal-Pinczon series.

The papers [1-3] are devoted to the study of simple $\mathfrak{sl}(2)$-modules and other closely related problems. In [2] there is given a classification of simple $\mathfrak{sl}(2)$-modules. The approach and classification of the present paper, which differ somewhat of those of [2], are probably more natural. As concerns the module of extensions, some results for the case of weight modules can be found in [4, 5].

1. Classification of Simple $\mathfrak{sl}(2)$-Modules. Let $K$ be an algebraically closed field of characteristic zero. Consider the Lie algebra $\mathfrak{sl}(2, K) = \langle Y, H, X \rangle | [X, Y] = H, [H, X] = 2X, [H, Y] = -2Y \rangle$, and let $U = U(\mathfrak{sl}(2))$ be its universal enveloping algebra, $C = H(H + 2) + 4XY$ the Casimir operator, $\delta$, half the sum of the positive roots, $K(H)$, the ring of rational functions, $\sigma \in \text{Aut} K(H)$, $\sigma(H) = H - 2$. Let $\tau \in K$ be a fixed scalar. We introduce the algebra $A = A_1 = U/U(C - \tau)$. From the Poincaré-Birkhoff-Witt theorem it follows that

$$A = (Y, H, X | XH = (H - 2)X, YH = (H + 2)Y, XY = \sigma(a)),$$

where $a = \frac{1}{4} (\tau - H^2 - 2H)$. (To simplify notation, the elements of $U$ and $A$ will be denoted by the same letters, with no danger of confusion.)

An immediate consequence of (1) is that $A = \bigoplus A_n$ is a graded ring, an integral domain, $\mathbb{N}$

and a free left and right $K[H]$-module; here $A_0 = K[H]$, $A_n = A_0 X^n$, $A_{-n} = A_0 Y^n$ for $n$, a positive integer, and $\pi : A \rightarrow A_0$ is the projection onto the first summand, which is a homomorphism of $K[H]$-bimodules. Let $K[H][X^\pm]$; $\sigma$ denote the ring of skew polynomials in the variables $X$ and $X^{-1}$ with coefficients from the field $K(H)$, i.e., $X^\pm a = \sigma^\pm (a) X^\pm$, for any $a \in K(H)$. Then $K[H][X^\pm]$; $\sigma$ is a noncommutative Euclidean ring with respect to the "length" map $\xi : \mathbb{Z} \times ((a_0 X^m + \ldots + a_n X^n) = m, m < \ldots < n, a_i \in K(H), a_0 a_n \neq 0$. Let $S = K[H] \setminus \{0\}$ be a closed multiplicative set and let $B = S^{-1} A$ be the corresponding localization of $A$. The algebra $A$ embeds naturally in $B$, and in what follows we identify it with its image under this embedding.

Proposition 1. $B \cong K(H)[X^\pm]$; $\sigma$.

For the proof it suffices to remark that $X^{-1} = a^{-1} Y$.

On a simple module the Casimir operator acts as a scalar [4]. Unless otherwise stipulated, we shall assume that this action is given by the fixed number $\tau \in K$ for all simple modules considered in this paper.

The module $V$ is called a weight module (generalized weight module) if $V = \bigoplus V_\lambda \implies \lambda \in K$, where $V_\lambda = \text{Ker}(H - \lambda) V$ is the component of $V$ of weight $\lambda$ (resp., $V = \bigoplus V^\lambda \implies \lambda \in K$, where $V^\lambda = \bigcup_{\lambda > \lambda} \text{Ker}(H - \lambda) V$).
The support \( \text{Supp} V \) of the weight (generalized weight) module \( V \) is defined to be the set of all \( \lambda \in K \), such that \( V_{\lambda} \neq 0 \) (resp., \( V_{\lambda} = 0 \)).

An immediate consequence of the graded decomposition of \( A \) is that the module \( S_{\tau, \theta} = A / A (\theta - 0) \), \( \theta \in K \), is a weight module, of finite length, its support equals \( \theta + 2\mathbb{Z} \), all its nontrivial weight components have dimension 1, and the module is generated by any element whose weight decomposition contains a nonzero component of weight \( \theta \). The module \( S_{\tau, \theta} \) is simple if and only if \( a(\theta + 2n) = 0 \) for all integers \( n \).

**Proposition 2.** Let \( \mathcal{I} \) be a nonzero left ideal of \( A \), let \( \alpha(H) \) be a generator of the ideal \( (\mathcal{I}) \), \( I_n = \mathcal{I} + A (H - 0)^n \), \( \theta \in K \), and \( n \) be a positive integer. Then:

1) \( \mathcal{I} + A (H - 0) \neq A \Leftrightarrow \theta \) is a root of the polynomial \( \alpha(H) \);
2) \( \theta \) is a root of multiplicity \( s \) of the polynomial \( \alpha(H) \), then \( I_n = I_m \) for all \( m \), \( n \geq \).

**Proof.** 1. \( I_1 \neq A \Leftrightarrow I_1 \) is a proper submodule of \( S_{\tau, \theta} \), where the bar denotes the epimorphism \( A \rightarrow S_{\tau, \theta} \) the support of the module \( I_1 \) does not contain \( \theta \Rightarrow \theta \) is a root of \( \alpha(H) \).

2. Let \( \alpha(H) = (H - 0)^{\beta}(H) \), \( \beta(0) \neq 0 \). Let us show that \( I_{n-1} = I_n \) for all \( n > s \). Consider the epimorphism \( A \rightarrow A (A (H - 0)^{n-1}, y \rightarrow \gamma \). There exists an element \( x = x_0 + x_1 + \ldots + x_t \in \mathcal{I} \), in the graded decomposition of which \( x_0 = \alpha(H) \). Set \( \psi(H) = (H - 0 - 2\beta) \ldots (H - 0)^{N} \). \( \varphi(H) = \beta(0)^{s-1}(\psi(0)^{s-1}(H - 0)^{s-1}\psi(H) \). Then \( \varphi(x) = (H - 0)^{s-1} \in \mathcal{I} \). Therefore, \( I_{n-1} \subseteq I_n \). The proposition is proved.

The classification of simple weight \( s\mathfrak{g}(2) \)-modules is well known [1, 6]. We, however, are interested mainly in non-weight modules.

**Proposition 3.** Let \( M \) be a simple, non-weight \( A \)-module. Then: 1) \( S_1 \) is a simple \( B \)-module that contains the \( A \)-module \( M \); 2) \( M \cong M_p = A / A \cap Bp \) for some prime element \( p \notin B \), which can be always assumed to belong to \( A \).

**Proof.** It suffices to remark that every simple \( B \)-module is of the form \( B / pB \), where \( p \) is prime. The proposition is proved.

**Proposition 4 (criteria for the simplicity of the module \( M_p \)).** Let \( p \notin B \) be a prime element, and let \( \alpha(H) \) be a generator of the ideal \( \pi (A \cap Bp) \). Then the following assertions are equivalent:

1) \( A \cap Bp \) is not a left maximal ideal of \( A \);
2) there is a \( 0 \neq \beta \in K \{H\} \), such that \( A \cap Bp + A\beta \neq A \);
3) there is a \( 0 \neq \alpha_p \in K \{H\} \), such that \( \pi (A \cap Bp\alpha_p^{-1}) \neq K \{H\} \).

Moreover, \( A \cap Bp + A (H - 0) \neq A \Leftrightarrow \theta \) is a root of the polynomial \( \alpha(H) \).

**Proof.** 1) \( \Rightarrow \) 2). Suppose \( A \cap Bp \) is strictly included in some left maximal ideal \( \mathfrak{I} \). Since \( Bp \) is a left maximal ideal in \( B \), \( S_1 \) is in \( B / pB \) and \( 0 \neq I / K \{H\} \beta \neq 0 \). Then \( A \cap Bp + A\beta \subseteq \mathfrak{I} \neq A \).

Now let \( \beta \) be as in 2). Then in view of the existence of the "length" function \( l \), the element \( \beta \) does not belong to \( A \cap Bp \) for \( l(\beta) = 0 \) and \( l(v) \geq l(\beta) > 0 \) for any \( v \in A \cap Bp \). Consequently, \( A \cap Bp \) is not a left maximal ideal.

2) \( \Rightarrow \) 3). \( A \cap Bp + A\beta \neq A \Leftrightarrow \) there exist \( 0 \neq \alpha_p \in K \{H\} \), and \( \lambda \in K \) such that \( \mathcal{I} = A \cap Bp + A\alpha_p (H - \lambda) \). But \( \mathcal{I} \cap L \cong A (A (H - \lambda) + A \cap Bp\alpha_p^{-1}) \), which is in view of Proposition 2 equivalent to the inclusion \( \alpha (A \cap Bp\alpha_p^{-1}) \subseteq A \alpha_p (H - \lambda) \). The proposition is proved.

2. Finite-Length Modules. **Proposition 5.** Any finitely-generated, generalized weight \( A \)-module \( L \) is a finite-length module.

**Proof.** Clearly, \( L \) is a quotient module of a finite direct sum of modules of the form \( L_n = A / A (H - 0)^n \), \( \theta \in K \), \( n \) a positive integer. Hence, it suffices to show that \( L_n \) is of finite length. If \( n = 1 \), then \( L_1 = S_{\tau, \theta} \) is indeed of finite length. Using the existence of the natural exact sequence \( 0 \rightarrow S_{\tau, \theta} \rightarrow L_n \rightarrow L_{n-1} \rightarrow 0 \), the proof can be completed by induction on \( n \). The proposition is proved.

**Proposition 6.** Let \( p \in B \) be prime and suppose \( J = A \cap Bp \) is not a left maximal ideal of \( A \). Then: