STABILITY OF WEIGHTED DIFFERENCE SCHEMES

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Estimates of stability of weighted differences in norms of Banach spaces are constructed. On the basis of these, corresponding estimates of stability in the norms of the spaces $C_0$ and $L_p$, $1 \leq p < \infty$, are obtained for difference schemes which approximate an initial-boundary value problem for the heat equation with boundary conditions of the first, second, and third kinds. In addition, the estimates of resolvents of difference elliptic operators in $C_0$ and $L_p$, $1 \leq p < \infty$, obtained in this article are used in an essential way.

1. In a family of Banach spaces $E_h$ which depend on some scalar or vector parameter $h$, we consider the weighted difference scheme

$$
\tau^{-1}|y(t_{k+1}) - y(t_k)| + \sigma A_h y(t_{k+1}) + (1 - \sigma) A_h y(t_k) = F(t_k),
$$

$$
\quad k = 0, 1, \ldots, T/\tau - 1,
$$

$$
y(t_0) = y_0,
$$

where $t_k = k\tau \in [0, T]$ is a discrete argument ($T$ is an upper bound of the variable $t_0$, and $\tau$ is the step of the discretization), $y(t_k)$ is a function of the discrete argument $t_k$ with values in $E_h$ which is a solution of the scheme, $A_h : E_h \to E_h$ is some bounded linear operator (for each fixed $h$), $\sigma$ is a numerical parameter called the weight of the scheme which may depend on $\tau$ and $h$, $y_0 \in E_h$ is the initial data of the scheme, and $F(t_k)$ is a known function of the discrete argument $t_k$ with values in $E_h$, which makes the scheme nonhomogeneous. Weighted difference schemes which approximate initial-boundary value problems for parabolic equations, for example, reduce to schemes of the type (1) if we write them in operator form.

Definition. We will call an operator $A_h : E_h \to E_h$ uniformly $v$-positive with angle $\varphi$, $0 < \varphi < \pi/2$, if for some given function $v(h)$ the bound $A_h$

$$
\|R(\lambda, A_h)\|_{E_h} \leq C_0 |\lambda|^{-\varphi},
$$

where $\rho_h(\varphi) = \{z : |\arg z| \leq \varphi\}$, $\|\lambda\| \geq \delta_0$, $\|h\| \leq h_0$

on the resolvent of $A_h$ holds for some nonnegative constants $C_0$, $\delta_0$, and $h_0$ which do not depend on $\lambda$ or $h$, where $\rho_h(\varphi)$ is a rhombus of the form

$$
\rho_h(\varphi) = \{z : |\arg z| \leq \varphi\} \cap \{z : |\arg(v(h) - z)| \leq \varphi\}.
$$

We note that the requirement of uniform $v$-positivity is somewhat more stringent than the well-known condition of uniform strong positivity [1].

THEOREM 1. If the operator $A_h$ in (1) satisfies the condition of uniform $v$-positivity with angle $\varphi_0$ for some $\varphi_0 \in (0, \pi/2]$, then for $\sigma \geq 1/2 - 1/\tau \nu (h)$ the following estimate of the stability of the scheme (1) holds:

$$
\|y(t_k)\|_{E_h} \leq C_1 \|y_0\|_{E_h}, \quad C_1 = \text{const}, \quad k = 0, 1, \ldots, T/\tau, \quad \|h\| \leq h_0.
$$
In addition, if it is possible to put the constant \( \delta_0 \) in (2) equal to zero, then the constant \( C_1 \) in (3) does not depend on \( T \), which means that (3) is satisfied for \( t_h \in [0, \infty) \), i.e., for \( k = 0, 1, \ldots \).

To prove this theorem, we first consider the case \( \delta_0 = 0 \) and establish that the operator \( \hat{A} \),

\[
\hat{A} = A_h | I + \sigma tA_h |^{-1}
\]

satisfies the \( (A_1) \)-condition of [2], from which the corresponding estimate of the stability of the scheme (1) follows. The extension to the case \( \delta_0 \neq 0 \) is trivial in the light of a remark made in [2].

2. We consider a differential operator of the form

\[
[Au](x) = -a(x) \frac{d^2 u(x)}{dx^2} + b(x) \frac{du(x)}{dx} + c(x) u(x), \quad x \in (0, 1)
\]

with variable coefficients \( a(x) \), \( b(x) \), and \( c(x) \), and defined on sufficiently smooth functions \( u(x), x \in [0, 1] \), which satisfy the boundary conditions \( du/dx - \beta_1 u = 0, x = 0 \), and \( du/dx + \beta_2 u = 0, x = 1 \), where \( \beta_1, \beta_2 \gg 0 \) are fixed parameters. For \( 0 < \beta_j < \infty \) the boundary condition is a condition of the third kind; for \( \beta_j = 0 \), of the second kind; and for \( \beta_j = \infty \), it passes formally into a condition of the first kind. In what follows, we will assume that \( \beta_j = \infty \). It is obvious that \( x_j = \xi h, \xi = 0, 1, \ldots, N \). We also define the mesh \( \omega_h \) by excluding the boundary mesh points \( x_0 \) and \( x_N \). We consider the following difference analog \( A_h \) of the operator \( A \):

\[
x_j = \xi h, \xi = 0, 1, \ldots, N. \quad \text{We also define the mesh } \omega_h \text{ by excluding the boundary mesh points } x_0 \text{ and } x_N. \quad \text{We consider the following difference analog } A_h \text{ of the operator } A:
\]

\[
\begin{align*}
[A_h y](x) &= \begin{cases} 
- \frac{2a(x)}{h} [(\Delta_{-} y)(x) + \beta_1 y(x)] + b(x) [(\Delta_{+} y)(x) + c(x) y(x)], & x = x_0 = 0, \\
- a(x) [(\Delta_{-} y)(x) + \beta_1 y(x)] + b(x) [(\Delta_{1} y)(x) + c(x) y(x)], & x \in \omega_h, \\
2\frac{a(x)}{h} [(\Delta_{-} y)(x) + \beta_1 y(x)] + b(x) [(\Delta_{-} y)(x) + c(x) y(x)], & x = x_N = 1,
\end{cases}
\end{align*}
\]

where \( y(x) \) is a function of the discrete argument \( x \in \omega_h \), the operator \( \Delta_2 \) corresponds to the second difference derivative

\[
[\Delta_2 y](x) = h^{-2} [y(x + h) - 2y(x) + y(x - h)],
\]

the operator \( \Delta_1 \) is the first central difference derivative

\[
[\Delta_1 y](x) = (2h)^{-1} [y(x + h) - y(x - h)],
\]

and the operators \( \Delta_+ \) and \( \Delta_- \) are the first one-sided difference derivatives

\[
\begin{align*}
[\Delta_{+} y](x) &= h^{-1} [y(x + h) - y(x)], \\
[\Delta_{-} y](x) &= h^{-1} [y(x) - y(x - h)].
\end{align*}
\]

If \( \beta_j = \infty \), then we will assume that \( [A_h y](x) = 0, x = x_{N+1} = j - 1 \). The operator \( A_h \) defined in (4) arises in difference schemes of the second order in \( h \) (see [3], for example).

We will take as \( E_h \) the space \( C_h \) of mesh functions \( y(x) \) defined on the mesh \( \omega_h \) with norm

\[
\| y(\cdot) \|_{C_h} = \max_{0 \leq j < N} | y(x_j) |
\]

or the space \( L_{p_h} \), \( 1 \leq p < \infty \), of mesh functions \( y(x), x \in \omega_h \) with the norm

\[
\| y(\cdot) \|_{L_{p_h}} = \left( \sum_{x \in \omega_h} | y(x_j) |^p h^{|j|} \right)^{1/p}.
\]

If \( \beta_j = \infty \), then we assume that the spaces \( C_h \) and \( L_{p_h} \) consist of functions \( y(x) \) which vanish for \( x = x_{N+1} = j - 1 \).

**THEOREM 2.** Suppose that the following assumptions concerning the coefficients \( a(x) \), \( b(x) \), and \( c(x) \) are fulfilled: the function \( a(x) \) is real, continuous, and positive on the