GENERALIZED GREEN’S MATRIX FOR LINEAR PULSE
BOUNDARY-VALUE PROBLEMS

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We establish an algebraic criterion of solvability, study the structure of general solutions of linear boundary-value problems for systems of differential equations with pulse effects, and construct the generalized Green’s matrix.

1. Introduction

Consider a linear system of differential equations

\[ \dot{x} = A(t)x + \varphi(t), \quad t \in [a, b], \quad t \neq \tau_i, \quad i \neq 1, p, \]

with pulse effects in the boundary conditions

\[ l(x) \equiv \sum_{i=1}^{p+1} l_i(x_i) = h. \]

Here, the \( n \times n \) matrix function \( A(t) \) is from \( C[a, b] \) and \( \varphi(t) \) is an \( n \)-vector function with discontinuities of the first kind at \( t = \tau_i, \quad i = 1, p \), i.e.,

\[ \varphi(t) = \varphi_i(t), \quad t \in [\tau_{i-1}, \tau_i], \quad i = 1, p+1, \quad \tau_0 = a, \quad \tau_{p+1} = b. \]

The linear functionals \( l_i(x_i) \) are defined in the \( n \)-vector space of continuously differentiable functions \( x_i(t) \) in \( [\tau_{i-1}, \tau_i], \quad i = 1, p+1 \), i.e.,

\[ l_i(x_i) : C^1[\tau_{i-1}, \tau_i] \to \mathbb{R}^m, \quad h \text{ is an arbitrary } m \text{-vector from } E^m. \]

Difference boundary-value problems obtained as a result depend on the individual form of linear functionals. Thus, multi-point boundary-value problems with \( p \)-points of pulse effects can be expressed in terms of the Stieltjes integral

\[ l_i(x_i) = \int_{\tau_{i-1}}^{\tau_i} [d\sigma_i(s)] C_i(s)x_i(s), \quad i = 1, p+1, \quad \tau_0 = a, \quad \tau_{p+1} = b, \]

where \( \sigma_i(s) = \text{diag} [\sigma_{i11}(s) \ldots \sigma_{i,nn}(s)] \) are diagonal \( m \times m \) matrices whose elements are functions of bounded variation in \( [\tau_{i-1}, \tau_i] \) and \( C_i(s) \) are \( m \times n \) matrices whose elements are functions with discontinuities of the first kind at \( t = \tau_i, \quad i = 1, p \).

We study the problem of finding a function

\[ x(t) = x_i(t), \quad t \in [\tau_{i-1}, \tau_i], \quad i = 1, p+1, \quad \tau_0 = a, \quad \tau_{p+1} = b. \]
with discontinuities of the first kind at \( t = \tau_i, \ i = 1, p + 1, \) such that the vector differentiable functions \( x_i(t) \) satisfy (1) and (2).

System (1), (2) is investigated, e.g., in [1], where boundary conditions are written in the form

\[
C_i x(\tau_i - 0) + D_i x(\tau_i + 0) = d_i, \quad i = 1, p, \tag{4}
\]

where \( C_i, B_i, \) and \( A_k \) are \( n \times n \) matrices and the pulse points \( \tau_k \) lie between the points \( t_k, \) i.e.,

\[
i_k < \tau_k < t_{k+1}, \quad k = 1, p. \tag{5}
\]

The homogeneous multi-point boundary-value problem corresponding to (1), (4), and (5) possesses a unique trivial solution.

In this paper, the problem posed above is solved by using the methods of half-inverse matrices and generalized Green's matrices. (For the theory of half-inverse matrices, see [2]; the theory of generalized Green's matrices is presented, e.g., in [3–5].)

In [6], the same methods are applied to system (1) with two-point boundary conditions and a single pulse effect in (5).

2. Principal Results

We introduce the following notation: \( \Phi(t) \) is the fundamental matrix of solutions of \( \dot{x} = A(t)x \) such that \( \Phi(a) = I_n (I_n \) are unit \( n \times n \) matrices), \( l_i(\Phi), \ i = 1, p + 1, \) are \( m \times n \) matrices, \( M = [l_1(\Phi) \ldots l_{p+1}(\Phi)] \) is an \( m \times (p + 1)n \) matrix, and \( M = [N_1 N_2 \ldots N_{p+1}]^T \) is an arbitrary half-inverse matrix of \( M, \) where \( N_i, \ i = 1, p + 1, \) are \( n \times m \) matrices.

1. The general solution of (1) has the form

\[
x(t) = \Phi(t)c + \bar{x}(t), \quad c \in \mathbb{R}^n, \tag{6}
\]

with the coordinate representation \( (\tau_0 = a, \ \tau_{p+1} = b) \)

\[
x_j(t) = \Phi(t)\Phi^{-1}(\tau_{j-1})x_j(\tau_{j-1}) + \bar{x}_j(t), \quad t \in [\tau_{j-1}, \tau_j], \quad j = 1, p + 1. \tag{7}
\]

The function \( \bar{x}(t) \) with discontinuities of the first kind is a particular solution of system (1). We choose it in the form

\[
\bar{x}(t) = \frac{1}{2} K(t, s) \varphi(s) ds, \quad s, t \in [a, b], \quad s, t \neq \tau_i, \quad i = 1, p. \tag{8}
\]

where

\[
K(t, s) = \frac{1}{2} \Phi(t) \Phi^{-1}(s) \text{sign}(t-s),
\]

\[
t, s \in [a, b], \quad t, s \neq \tau_i, \quad i = 1, p.
\]