ON KOLMOGOROV WIDTHS OF CLASSES $B_{p,\theta}^r$ OF PERIODIC FUNCTIONS OF MANY VARIABLES WITH LOW SMOOTHNESS IN THE SPACE $L_q$

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We study the Kolmogorov widths of Besov classes $B_{p,\theta}^r$ of periodic functions of many variables with low smoothness in the space $L_q$, $1 < q < \infty$. We also investigate the behavior of widths of such classes with critical indices of smoothness.

In studying the Kolmogorov widths $d_n(W^r_{1,q}, L_q)$ of the classes $W^r_1$ of periodic functions of one variable for $2 < q < \infty$, $1 - 1/q < r < 1$, Kashin [1] established that their behavior in the case of $r \leq 1$ (low smoothness) significantly differs from that in the case of $r > 1$ (high smoothness). Kulanin [2] generalized this result to the case of the classes $W^r_p$ and obtained upper and lower order estimates for the classes $W^r_p$ of functions of many variables [3] under the assumption that $r \in R^m$, $r = (r_1, \ldots, r_1)$, $r_1 > 0$. Later, Galeev [4] studied the Kolmogorov widths of the classes $W^r_p$ of functions of many variables with low smoothness in the case of $r = (r_1, \ldots, r_m)$, $r_1 > 0$, $j = \overline{1,m}$; he improved the estimates obtained in [3] and established order estimates for the Kolmogorov widths of the classes $H^r_p$ of functions of many variables [5].

In this paper, we study the behavior of the widths $d_M(B_{p,\theta}^r, L_q)$ for $1 < p < 2 < q < \infty$ and $2 < p < q < \infty$. Note that, for the same values of the parameters $p$, $q$, and $\theta$, the order of $d_M(B_{p,\theta}^r, L_q)$ in the case of the classes $B_{p,\theta}^r$ with high smoothness was established in [6]; the corresponding bibliography is also presented therein.

We develop the methods used in the papers mentioned above and obtain order estimates for the widths of classes which may be either narrower than $W^r_p$ or intermediate between $W^r_p$ and $H^r_p$ (for $\theta = \infty$, we have $H^r_p = B_{p,\infty}^r$).

Furthermore, we improve the well-known upper estimates for the classes $H^r_p$ [5] and establish bilateral estimates of the widths $d_M(B_{p,\theta}^r, L_q)$ in the case of “critical indices” of smoothness.

Below, we use the definitions and notation from [6–8]. It is also assumed that the condition

$$\int_{-\pi}^{\pi} f(x) dx_j = 0, \quad j = \overline{1,m},$$

is satisfied. For convenience, we present several known statements and notation that will be used in what follows.

Let $R^m$ denote the Euclidean space with elements

$$x = (x_1, \ldots, x_m), \quad (x, y) = x_1 y_1 + \ldots + x_m y_m,$$

and let $L_p(\pi_m)$ be the space of periodic functions $f(x) = f(x_1, \ldots, x_m)$ given on...
\[\pi_m = \prod_{j=1}^{m} [-\pi, \pi]\]

with the finite norm

\[\|f\|_p = \left(\frac{1}{(2\pi)^{-m}} \int_{\pi_m} |f(x)|^p \, dx\right)^{1/p}, \quad p \in (1, \infty)\]

For \(s = (s_1, \ldots, s_m), s_j \in \mathbb{N}, j = 1, m,\) we set

\[p(s) = \{k: k = (k_1, \ldots, k_m), \quad 2^{j-1} \leq |k_j| < 2^j, \quad j = 1, m\}, \]

\[\delta_j(f, x) = \sum_{k \in p(s)} \hat{f}(k) e^{i(k \cdot x)},\]

\[\hat{f}(k) = (2\pi)^{-m} \int_{\pi_m} f(t) e^{-i(k \cdot t)} \, dt.\]

Let \(l_p^m\) denote the space with the norm

\[\|x\|_{l_p^m} = \begin{cases} \left(\sum_{j=1}^{m} |x_j|^p\right)^{1/p}, & \ 1 \leq p < \infty, \\ \max_{1 \leq i \leq m} |x_i|, & \quad p = \infty, \end{cases}\]

and let \(B_p^m\) be a unit ball in \(l_p^m\). If \(P\) is a subset of an integer lattice, then \(|P|\) denotes the number of elements of \(P\).

**Lemma A [9].** Let

\[f(x) = \sum_{s \in S} \delta_s(f, x), \quad s \in \mathbb{N}^m, \quad 1 < p < \infty.\]

Then

\[|S|^{(1/2 - 1/p)} \left(\sum_{s \in S} \|\delta_s(f, x)\|_p^p\right)^{1/p} \ll \|f\|_p \ll |S|^{(1/2 - 1/p) + \left(\sum_{s \in S} \|\delta_s(f, x)\|_p^p\right)^{1/p}},\]

where \(a_- = \min\{a, 0\}, \ a_+ = \max\{a, 0\} \).

**Lemma B [10, p. 25].** Suppose that \(1 \leq p < q < \infty\) and \(f \in L_q(\pi_m)\) are given. Then

\[\|f\|_q \ll \left(\sum_s \left(\|\delta_s(f, x)\|_p 2^\|s\|_{(1/p-1/q)}\right)^q\right)^{1/q},\]

where \(\|s\|_1 = s_1 + \ldots + s_m\).