We establish conditions under which solutions of some systems of differential equations are bounded and study their asymptotic properties.

Consider a system of equations

\[
\begin{align*}
\frac{dx_1}{dt} &= X_1(x_1, \ldots, x_n), \\
& \quad \text{\ldots} \\
\frac{dx_n}{dt} &= X_n(x_1, \ldots, x_n),
\end{align*}
\]

where \( X_1, \ldots, X_n \) are continuous functions. In [1], one can find the conditions (Theorem 4) under which the asymptotic stability of system (1) is attained for \( \dot{V} \leq 0 \), where \( V \) is a positive definite function (unlike the well-known Lyapunov criterion, where the inequality \( \dot{V} < 0 \) is required).

In this work, we suggest criteria that enable one to analyze the behavior of the solutions of some systems of differential equations with an arbitrary sign of the derivative \( \dot{V} \).

**Definition 1.** A positive definite function \( V(x_1, \ldots, x_n) \) is called infinitely large if, for any positive \( A \), one can indicate a number \( N \) such that \( V(x_1, \ldots, x_n) > A \) whenever \( \sum_{i=1}^{n} x_i^2 > N \).

**Definition 2.** Solutions of system (1) are called bounded if, for any initial point, one can find a sphere with finite radius that contains the corresponding trajectory for all \( t > t_0 \) where \( t_0 \) is initial time.

**Theorem 1.** In order that the solutions of system (1) be bounded for arbitrary initial perturbations, it is sufficient that there exist an infinitely large positive definite function \( V \) whose derivative satisfies the inequalities \( \dot{V} \geq 0 \) inside \( K \) and \( \dot{V} < 0 \) outside \( K \), where \( K \) is a bounded set of values of the variables \( x_1, \ldots, x_n \).

**Proof.** Since the set \( K \) is bounded by the condition of Theorem 1, one can find a sphere \( \eta \) that encloses the entire domain \( K \). Let

\[ \max_{\eta} V = l. \]

We also choose a sphere \( \varepsilon \) that encloses \( \eta \) and satisfies the following condition:

\[ \min_{\varepsilon} V = L, \]

where \( L > l \). It is clear that, in view of Definition 1, this relation can always be satisfied.
Let us show that if the motion starts at a point \( M_0 \in I_\eta \) (where \( I_\eta \) is the set of points lying inside the sphere \( \eta \)), then the trajectory never leaves the sphere \( \varepsilon \).

First, we assume that \( M_0 \in K \). In this case, we have \( \dot{V} \geq 0 \). Therefore, \( V \) may increase up to the value \( l \) because the domain \( K \) lies inside the sphere \( \eta \). As soon as the trajectory leaves \( K \), the function \( V \) becomes decreasing since \( \dot{V} < 0 \) outside \( K \).

If the motion starts at a point \( M_0 \in I_\eta \setminus K \), then \( V \) decreases. In this case, the trajectory may either asymptotically approach zero (by the Lyapunov criterion of asymptotic stability) or enter the domain \( K \). The latter possibility is discussed above.

Thus, whenever the motion starts inside the sphere \( \eta \), we have

\[
V \leq l
\]

for all \( t \geq t_0 \). If we assume that a trajectory leaves the sphere \( \varepsilon \), then, according to (3), at a point where this trajectory crosses \( \varepsilon \), we can write

\[
V \geq L
\]

but this contradicts (4) since \( L > l \). Therefore, in the case under consideration, the trajectory remains inside the sphere \( \varepsilon \) and, hence, the solutions of system (1) are bounded.

**Example 1.** Consider the system

\[
\begin{align*}
\dot{x} &= x - x^3 - 2y, \\
\dot{y} &= 2x - y - y^3.
\end{align*}
\]

(6)

We investigate this system by using the infinitely large positive definite function \( V = (x^2 + y^2)/2 \). By virtue of Eqs. (6), we have

\[
\dot{V} = -x^4 + x^2 - y^4 - y^2.
\]

Function (7) is not negative definite because the inequality \( \dot{V} > 0 \) is possible provided that \( x^2 > x^4 + y^2 + y^4 \). At the same time, it is obvious that \( \dot{V} \) is always negative outside the square \( |x| = 1, |y| = 1 \). Thus, the domain, where \( \dot{V} \geq 0 \), is bounded and, therefore, the solutions of system (6) are bounded under any initial perturbations.

Consider a system

\[
\begin{align*}
\dot{x} &= X(x, y), \\
\dot{y} &= Y(x, y).
\end{align*}
\]

(8)

**Theorem 2.** Assume that system (8) satisfies the conditions of Theorem 1, has no closed orbits in the entire phase plane, and that the origin is the sole singular point of this system. Then system (8) is asymptotically stable under any initial perturbations.

**Proof.** Suppose that the conditions of Theorem 1 are satisfied. This means that the trajectory always remains inside a certain sphere. It is well known [3] that, in this case, it either approaches a singular point, or approaches a closed trajectory, or is a closed orbit itself. However, according to the conditions of Theorem 2, there are no closed trajectories and singular points except the origin and, therefore, the origin is the only point that attracts all trajectories.