
NUMERICAL ANALYTIC METHOD FOR THE CASE OF SINGULAR MATRICES IN BOUNDARY CONDITIONS

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A modification of the numerical analytic method of successive approximations is proposed for investigating the existence and constructing solutions of systems of nonlinear ordinary differential equations with two-point linear boundary conditions. The method is tailored to the case where the matrices contained in boundary conditions are singular, but their linear combination is nonsingular.

Introduction. In the present article, the numerical analytic method of successive approximations [1] is extended to investigating the existence and constructing approximate solutions of the nonlinear differential equations

\[ x = f(t, x), \quad x, f \in \mathbb{R}^n, \quad t \in [0, T], \tag{1} \]

considered with the two-point nonseparable boundary conditions

\[ Ax(0) +Cx(T) = d, \tag{2} \]

where $x, f, d$ are points of the $n$-dimensional Euclidean space $\mathbb{R}^n$, $A, C$, are constant $n \times n$ matrices such that for some fixed real numbers $k_1, k_2$, $\det (k_1A + k_2C) \neq 0$. Since the matrices $A$ and $C$ may be singular, the numerical analytic scheme developed in [1] for problem (1), (2) does not apply. It is shown that for problem (1), (2) a sequence of functions dependent on a parameter can be constructed which converges to the sought solution for certain values of the parameter. From the properties of the constructed approximate solutions, a conclusion is drawn on the solvability of the boundary-value problems under consideration.

1. Choosing the Type, and the Convergence of Successive Approximations. Let the right-hand side of Eq. (1) be a function

\[ f(t, x): [0, T] \times D \to \mathbb{R}^n, \tag{3} \]

where $D$ is a closed bounded region in the space $\mathbb{R}^n$, and let in the indicated domain this function be continuous, bounded by the vector $M = (M_1, \ldots, M_n)$, $M_i \geq 0$, and satisfy the Lipschitz condition with a matrix $K = \{K_{ij} \geq 0, i, j = 1, 2, \ldots, n\}$:

$$|f(t, x)| \leq M, |f(t, x') - f(t, x)| \leq K|x' - x|,$$

where $|f(t, x)| = (|f_1(t, x)|, \ldots, |f_n(t, x)|)$ and the vector inequalities are understood componentwise. The class of boundary-value problems under consideration will be restricted to those for which the parameters $M$, $K$, $A$, $C$, $d$, $k_1$, $k_2$, and the domain indicated in (3) satisfy the following additional conditions:

1) the set $D_\delta$ of points $x_0 \in \mathbb{R}^n$ such that the points $x_0 + k_iH[d - (A + C)x_0] = z_0(x_0)$ are contained in $D$ together with their $\beta$-neighborhoods, where $H = (k_1A + k_2C)^{-1}$, $\beta = \frac{T}{n}M + \beta_1(x_0)$, $\beta_1(x_0) = |(k_2 - k_1)H[d - (A + C)x_0]|$, is not empty:

$$D_\delta \neq \emptyset;$$

2) the largest eigenvalue $\lambda(Q)$ of the matrix $Q = \frac{T}{n}K$ is less than unity:

$$\lambda(Q) < 1.$$  

In order to construct a sequence of functions satisfying boundary conditions (2), consider a sequence of the form

$$x_m(t, x_0) = x_0 + k_iHd(x_0) + \int_0^t \left[ f(t, x_{m-1}(t, x_0)) - \frac{1}{T} \int_0^T f(s, x_{m-1}(s, x_0)) ds \right] dt + \alpha[k_iT + (k_2 - k_1)t], \quad m = 1, 2, \ldots, \quad x_0(t, x_0) = x_0 + \alpha k_iT,$$

regarding $x_0 = (x_{0_1}, \ldots, x_{0_n})$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ as parameters.

Choose the parameter $\alpha$ so that functions (7) will satisfy boundary conditions (2) for all $m = 1, 2, \ldots$ and an arbitrary value of the other parameter $x_0 \in D_\delta$. Substituting (7) into (2), we obtain a system of algebraic equations for $\alpha$, whence

$$\alpha = \frac{1}{T}H[d - (A + C)x_0].$$

Therefore, all the terms of the sequence of functions

$$x_m(t, x_0) = x_0 + k_iHd(x_0) + \int_0^t \left[ f(t, x_{m-1}(t, x_0)) - \frac{1}{T} \int_0^T f(s, x_{m-1}(s, x_0)) ds \right] dt + \frac{1}{T}H[k_2 - k_1]d(x_0),$$

where $d(x_0) = d - (A + C)x_0$, satisfy boundary conditions (2) for an arbitrary $x_0$.

The following assertion on the convergence of the successive approximations $x_m(t, x_0)$ of the form (8) is valid.

**Theorem 1.** Let the right-hand side $f(t, x)$ of system (1) be continuous in the domain indicated in (3), and let conditions (4)-(6) be met.

Then a sequence of functions $x_m(t, x_0)$ of the form (8) uniformly converges in the region $(t, x_0) \in [0, T] \times D_\delta$ to its limit function $x^*(t, x_0)$ as $m \to \infty$. At that, $x^*(t, x_0)$ is a solution of the integral equation

$$x(t) = z_0(x_0) + \int_0^t \left[ f(t, x(t)) - \frac{1}{T} \int_0^T f(s, x(s)) ds \right] dt + \frac{1}{T}H[k_2 - k_1]d(x_0) dt$$

$$x^*(0, x_0) = x_0 + k_iHd(x_0) = z_0(x_0),$$

and, furthermore, $x^*(t, x_0)$ satisfies boundary conditions (2), i.e., is a solution of the perturbed boundary-value problem.