GROUPS ALL OF WHOSE INFINITE ABELIAN pd-SUBGROUPS ARE NORMAL

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The author studies groups in which any infinite Abelian pd-subgroup (p is a prime) is normal, on the assumption that the group indeed contains such subgroups (IHp-groups). Necessary and sufficient conditions are established for a group to be an IHp-group. Relationships are established between the class of IHp-groups and the class of groups in which all infinite Abelian subgroups are normal, and the class of groups in which all pd-subgroups are normal.

One of the most fruitful branches of modern group-theoretical research is the study of groups in which certain subgroups or systems of subgroups satisfy certain restrictive conditions. Work in this field has produced many important classes of groups, among them what are known as Hamiltonian groups - non-Abelian groups all of whose subgroups are normal.

Intermediate between the Abelian and Hamiltonian groups, on the one hand, and the simple groups, on the other, are all other groups that are "saturated" to a greater or lesser degree with normal subgroups. Retaining normality of subgroups as a restrictive condition but narrowing down the system of subgroups that have to satisfy the condition, one obtained different classes of groups which depart to a greater or lesser degree from Hamiltonian groups.

A comprehensive program for studying groups with various systems of normal subgroups has been formulated by Chernikov [1-4].

The results of this paper constitute a natural sequel to the studies of groups with various systems of infinite normal subgroups, initiated by Chernikov, and to the investigations of groups with various systems of normal pd-subgroups (p is a prime) that has been occupying me for the past few years [5-9].

The class of IH-groups was studied in [1, 4]. An infinite non-Abelian group that has at least one infinite normal Abelian pd-subgroup is called an IH-group if all its infinite Abelian subgroups are normal. In [5] I studied groups in which all pd-subgroups are normal, for some prime p; such groups are known as pdI-groups. In this paper I will be concerned with infinite non-Abelian groups that contain infinite Abelian pd-subgroups for some prime p and all such subgroups are normal. These groups will be called IHp-groups.

**Lemma 1.** Any IHp-group contains a normal subgroup of order p.

**Proof.** Let G be a periodic IHp-group and A = Ap x Ap' an infinite normal Abelian pd-subgroup of G. If Ap' is infinite, then for any a ∈ Ap we have ⟨a, Ap⟩ < G and therefore, ⟨a⟩ < G. If Ap' is finite, then Ap will be infinite. If it contains a quasicyclic subgroup P, then P < G and furthermore ⟨a⟩ < G for any a ∈ P. If Ap does not contain a quasicyclic subgroup, it does not satisfy the minimum condition. In that case, for any element a ∈ Ap, one can find in Ap two infinite subgroups B1 and B2 such that B1 ∩ B2 = 1 and ⟨a⟩ ∩ B1, B2 = 1. Then ⟨a, B1⟩ ∩ ⟨a, B2⟩ = ⟨a⟩ < G.

Now let G be a nonperiodic IHp-group and A an infinite normal Abelian pd-subgroup. By what we have proved, A contains a normal cyclic p-subgroup if it is periodic. But if
A is not periodic, then \( A \supset \langle a, x \rangle \), where \(|a| = p\), \(|x| = \infty\). Then \( \langle a, x \rangle \triangleleft G \) and therefore, \( \langle a \rangle \triangleleft G \). QED.

**COROLLARY 1.** Any nonperiodic IH\( _p \)-group contains an infinite cyclic normal subgroup.

**THEOREM 1.** A non-Abelian periodic group \( G \) that contains an infinite Abelian pd-subgroup is an IH\( _p \)-group but not an IH-group if and only if it has a proper normal subgroup \( C \) satisfying the following conditions:

1) \( C \) is an Abelian group or an IH-group;
2) \( C = C_{G}(a) \), where \(|a| = p\) and \( \langle a \rangle \triangleleft G \);
3) \( C \) contains any infinite Abelian pd-subgroup of \( G \);
4) any infinite Abelian subgroup of \( C \) is normal in \( G \);
5) \( G \) has an infinite Abelian p'-subgroup not contained in \( C \).

**Proof.** Sufficiency. This is obvious. We prove necessity.

Let \( G \) be a periodic IH\( _p \)-group that is not an IH-group. By Lemma 1, it contains a normal subgroup \( \langle a \rangle \) of order \( p \). Then the subgroup \( C_{G}(a) = C \) is Abelian or an IH-group, and all its infinite Abelian subgroups are normal in \( G \). Indeed, if \( B \) is an infinite Abelian subgroup of \( C \) and \( p \not\in \pi(B) \), then \( B \triangleleft G \) by the definition of an IH\( _p \)-group. If \( p \in \pi(B) \), then \( \langle a, B \rangle \triangleleft G \) and therefore, \( B \triangleleft G \).

Let \( H \) be an arbitrary infinite Abelian pd-subgroup of \( G \). If \( a \in H \), then \( H \subseteq C \). If \( a \not\in H \), then \( [a, H] = \langle a \rangle \cap H = 1 \), since \( H \triangleleft G \). Thus, in this case too \( H \subseteq C \). Consequently, \( C \) satisfies conditions 1-5 and the proof is complete.

**COROLLARY 2.** Any periodic IH\( _2 \)-group is an IH-group.

We will now examine nonperiodic IH\( _p \)-groups.

**THEOREM 2.** A nonperiodic non-Abelian group \( G \) that contains an infinite Abelian pd-subgroup is an IH\( _p \)-group if and only if it is of one of the following types:

1) \( G \) is a nonperiodic IH-group containing an infinite Abelian pd-subgroup;
2) \( G \) is a nonperiodic non-Abelian pdI-group;
3) \( |C_{p}| = p \neq 2 \), \( G_{p} \subseteq Z(G) \) and \( G/G_{p} \) is a nonperiodic IH-group;
4) \( p = 2 \), \( a \) is the only central involution in \( G \) contained in any infinite Abelian 2d-subgroup of \( G \), and \( G/\langle a \rangle \) is an IH-group;
5) \( p = 2 \), \( G = C_{b} \), where \( C \) is a nonperiodic Abelian pd-subgroup, \( b^{n} = 1 \), \( b^{-1}cb = c^{-1} \) for any \( c \in C \) and the center of \( G \) is infinite;
6) \( p \neq 2 \), \( G = \langle a \rangle \times C_{1} \langle b \rangle \), where \(|a| = |C_{p}| = p\), \( C_{1} \) is a nonperiodic Abelian group, \( \langle a, C_{1} \rangle \) is a nonperiodic pdI-group which is the centralizer of any infinite cyclic normal subgroup of \( G \), \( b^{2n} = 1 \), \( n \geq 1 \) and \( b^{2} \not\in C_{1} \); any noncyclic Abelian pd-subgroup of \( G \) is contained in \( C_{G}<a, x> \), where \(|x| = \infty\) and \( \langle x \rangle \triangleleft G \) and any pd-subgroup of \( C_{G}<a, x> \) is normal in \( G \);
7) \( G = ((a) \times A \langle b \rangle \langle d \rangle) \), where \(|a| = p \neq 2 \), \( \langle a \rangle = G_{p} \triangleleft G \), \( C_{a} \langle a \rangle = \langle a \rangle \times A \langle b \rangle \) and \( A \langle b \rangle \triangleleft G \), \( A_{<b>} \) is a nonperiodic IH-group and all subgroups of \( A \) are normal in \( G \), \( d^{n} \in C_{a} \langle a \rangle \) and \( n \) divides \( p - 1 \).

**Proof.** Each of the groups described in the theorem is an IH\( _p \)-group. Lemmas 2-8 comprise the proof that the conditions are necessary.

**LEMMA 2.** If the center of a nonperiodic IH\( _p \)-group \( G \) contains a noncyclic subgroup of order \( p^{2} \), then \( p = 2 \) and \( G \) is an IH-group.

**Proof.** Let \( G \) be a nonperiodic IH\( _p \)-group and \( Z(G) \supseteq \langle a_{3} \rangle \times \langle a_{4} \rangle \), where \(|a_{3}| = |a_{4}| = p\). Let \( A \) be an arbitrary infinite Abelian subgroup of \( G \). If \( p \in \pi(A) \), then \( A \triangleleft G \) by the definition of an IH\( _p \)-group. If \( p \not\in \pi(A) \), then \( \langle a_{3}, A \rangle \cap \langle a_{4}, A \rangle = A \triangleleft G \). Thus \( G \) is an IH-group. Since the center of a nonperiodic IH-group is finite and does not contain elements of order greater than two \([4]\), it follows that \( p = 2 \), proving the lemma.

**LEMMA 3.** If all infinite cyclic subgroups of a nonperiodic non-Abelian group \( G \) are normal, then \( G = C_{b} \), where \( C \) is a nonperiodic Abelian group, \( b^{n} = 1 \) and \( b^{-1}cb = c^{-1} \) for any \( c \in C \).