The differential equation $\xi = G(t) x$, $0 \leq t \leq T$, where $G$ and $G(t)$ are bounded linear operators, is examined in the complex Hilbert space conditions are found under which this equation is reducible to the canonical equation by a change of variables.

A linear differential equation in the complex Hilbert space $H$ has canonical structure if it is representable in the form

$$Jx = B(t)x, \quad 0 \leq t \leq T,$$

where $B(t)$ is a bounded linear Hermitian operator, $J$ is a bounded linear skew-Hermitian operator ($J^2 = -I$); the dot means a derivative of the function $x(t)$ with values in $H$ with respect to $t$ (see, for example, [1]).

We will consider equations of the more general type

$$G(t)x = \xi,$$  \hfill (1)

where $G(t)$ has the same properties as $B(t)$, and $\xi$ is a bounded linear operator in $H$.

We assume that $\xi$ has a bounded inverse operator. Then $\dot{x} = \xi^{-1} G(t)x$. We make an invertible change of variables for the unknown function $x = D^{-1}y$. The equation then takes the form

$$\dot{y} = D\xi^{-1}G(t)D^{-1}y.$$  \hfill (2)

We pose the following problem: what properties should the operator $\xi$, possess for there to exist an operator $D$ such that Eq. (2) would have canonical structure for any Hermitian operator operator $G(t)$.

We first prove an auxiliary statement.

**Lemma.** For the operator $ABC$ to be Hermitian for any Hermitian operator $B$, it is sufficient, and if $\ker C^* = 0$ or $\ker A = 0$ then also necessary, that the equality $A = \alpha C^*$, where $\alpha$ is a real number, is satisfied.

That the condition $A = \alpha C^*$ is sufficient is checked directly.

That it is necessary is proved as follows. Let

$$(ABC)^* = ABC$$  \hfill (3)

for any Hermitian operator $B$ and $\ker C^* = 0$. Choose for $B$ a one-dimensional Hermitian operator $Bx = (x, e)e$, where $e$ is an arbitrary nonzero element. Then $(Cx, e)Ae = (A^*x, e)C^*e$ for any $x \in H$. Set $x = C^*e$. Then we obtain $Ae = k(e)C^*e$, where $k(e) = (C^*e, Ae)[(C^*e, C^*e)]^{-1}$.

Note that $k(e) = k(\lambda e)$ for any $\lambda \in \mathbb{C}$. Suppose furthermore that the elements $f$ and $e$ are linearly independent. Form the condition $\ker C^* = 0$, the elements $C^*f$ and $C^*e$ are linearly independent. Consider

$$Af = A(f - e) + Ae = k(f - e)C^*(f - e) + k(e)C^*e = |k(e) - k(f - e)|C^*e + k(f - e)C^*f.$$  

On the other hand, $Af = k(f)C^*f$. By the linear independence of $C^*f$ and $C^*e$, we obtain that $k(e) - k(f - e) = 0$ and $k(f - e) = k(f)$. Thus $k(e) = k(f)$, and hence $k(e)$ is a constant, $k(f) = \alpha$. Then $A = \alpha C^*$. Substituting into Eq. (3) the operator $A/\alpha$ for the operator $C^*$, and the operator $\alpha C$ for the operator $A^*$, we obtain $\alpha = \alpha$, i.e., $\alpha$ is a real number.

If the condition $\ker A = 0$ is satisfied, then analogous arguments are carried out for $l(e) = (Ae, C^*e) [\langle Ae, Ae \rangle]^{-1}$. The lemma is proved.

We now return to the posed problem. Equation (2) has canonical structure only if there exists an operator $J$ such that the operator $JD\mathcal{E}^{-1}G(t)D^{-1}$ is Hermitian for any Hermitian operator $G(t)$. By the lemma, this condition holds if there exists a real constant $\alpha$ such that $JD\mathcal{E}^{-1} = \alpha(D^{-1})^*$, or

$$\mathcal{E} = \frac{1}{\alpha} D^*JD.$$  \hfill (4)

From this representation and the properties of the operator $J$, it follows that the operator $\mathcal{E}$ is skew-Hermitian.

The inverse statement is also valid: if the operator is skew-Hermitian, then it is representable in the form of Eq. (4). This is so since the operator $i\mathcal{E}$ is Hermitian and hence admits a spectral representation $i\mathcal{E} = \int \lambda dE_{\lambda}$.

The operator $i\mathcal{E}$ is invertible by assumption; hence zero is a regular point and its spectrum splits into two closed non-intersecting sets $\sigma_+$ and $\sigma_-$ which lie on the half-axes $(0; +\infty)$ and $(-\infty; 0)$, respectively. Consider the operators

$$J = -i \int_{-\infty}^{\infty} \chi(\lambda) dE_{\lambda} \text{ and } D = \int_{-\infty}^{\infty} \psi(\lambda) dE_{\lambda},$$

where

$$\chi(\lambda) = \text{sign } \lambda, \quad \psi(\lambda) = \begin{cases} \sqrt{\lambda}, & \text{for } \lambda > 0, \\ i\sqrt{\lambda}, & \text{for } \lambda < 0. \end{cases}$$

Then $J^* = -J$ and $J^2 = -I$. Furthermore, $D^* = \int_{-\infty}^{\infty} \psi_1(\lambda) dE_{\lambda}$, where $\psi_1(\lambda) = \overline{\psi(\lambda)}$. We compute

$$D^*JD = -i \int_{-\infty}^{\infty} \psi_1(\lambda) \text{sign } \lambda \psi(\lambda) dE_{\lambda} = \int_{-\infty}^{0} -i \sqrt{\lambda} \frac{i^2}{\sqrt{\lambda}} dE_{\lambda} =$$

$$+ \int_{0}^{\infty} \sqrt{\lambda} (\lambda - i) \sqrt{\frac{i^2}{\lambda}} dE_{\lambda} = -i \int_{-\infty}^{0} |\lambda| dE_{\lambda} - i \int_{0}^{\infty} |\lambda| dE_{\lambda} = \mathcal{E}.$$  

Thus the following theorem holds.

**THEOREM 1.** For Eq. (1) to be reducible by a standard transformation $x = D^{-1}y$ for any Hermitian operator $G(t)$ to Eq. (2), which has canonical structure, it is necessary and sufficient that the operator $\mathcal{E}$ be skew-Hermitian.

We call an equation of the form (1), where $\mathcal{E}$ is skew-Hermitian and $G(t)$ is a Hermitian operator, an implicit canonical equation.

We now examine the case where in the implicit canonical equation the operator $\mathcal{E}$ has a nonzero kernel $N = \text{Ker } \mathcal{E}$ and a closed range. Since $i\mathcal{E}$ is a Hermitian operator, the whole space splits into an orthogonal sum of subspaces $H = N \oplus N_1$, where $N_1$ is the range of the operator $i\mathcal{E}$. We denote by $P$ the orthogonal projection operator onto the subspace $N$. We substitute into Eq. (1) the value $x = Px + (I - P)x$:

$$\mathcal{E}(I - P)x = G(t)(Px + G(t)(I - P)x).$$  \hfill (5)

Apply the operator $P$, which commutes with $\mathcal{E}$, to both sides. Then $PG(t)Px + PG(t)(I - P)x = 0$.

Suppose that the Hermitian operator $P(t) = PG(t)P$ acting in the subspace $N$ has a bounded inverse. From the last equation we find that $Px = -F^{-1}(t)PG(t)(I - P)x$. Substituting this expression into Eq. (5), we obtain

$$\mathcal{E}(I - P)x = -G(t)F^{-1}(t)PG(t)(I - P)x + G(t)(I - P)x.$$  

Applying the operator $I - P$ to both sides and using $(I - P)\mathcal{E} = \mathcal{E}(I - P)$, we obtain

$$\mathcal{E}(I - P)x = -(I - P)G(t)F^{-1}(t)PG(t)(I - P)x + (I - P)G(t)(I - P)x.$$  

If we denote $(I - P)x = y$, then we obtain the equation

$$\mathcal{E}y = C(t)y$$  \hfill (6)

in the subspace $N_1$. In this subspace the skew-Hermitian operator $\mathcal{E}$ is invertible, and the operator $C(t) = -(I - P)G(t)F^{-1}(t)PG(t)(I - P) + (I - P)G(t)(I - P)$ is Hermitian. Indeed,