POINTS OF STRONG SUMMABILITY OF FOURIER SERIES

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The points of a function \( f \in L \) at which there are given estimates of the rate of convergence to zero of the strong arithmetic means of its Fourier series and the trigonometrically conjugate series are characterized.

Suppose \( f \) is a \( 2\pi \)-periodic, \( p \)-th power integrable \( (p \geq 1) \) function on \( \Delta = [-\pi, \pi] \), \( f \in L_p \), and \( S_n(f; x) \) is the partial sum of order \( n \) of its Fourier series.

The concept of the strong summability of Fourier series was introduced by Hardy and Littlewood [1]. We say that the Fourier series of a function \( f \in L_1 \) is \((H, q)\)-summable at a point \( x \) to \( f(x) \) if

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |f(x) - S_n(f; x)|^q = 0, \quad q > 0.
\]

In [2] Hardy and Littlewood showed that if \( f \in L_p, \ p > 1 \), then at each Lebesgue point \( x \) of degree \( p \), i.e., at a point \( x \) where

\[
\frac{1}{\delta} \int_{\delta}^{h} |f(x \pm t) - f(x)|^p \, dt = o(1),
\]

the equality (1) holds for any \( q > 0 \). Thus, the Fourier series of a function \( f \in L_p, \ p > 1 \), is \((H, q)\)-summable almost everywhere. They also showed there exists a function \( f \in L_1 \) whose Fourier series need not be \((H, q)\)-summable at a Lebesgue point for any \( q > 0 \).

Hardy and Littlewood posed this problem: Does (1) hold almost everywhere if \( f \in L_1 \)? This problem was solved affirmatively for \( q = 2 \) by Marcinkiewicz [3], and for any \( q > 0 \) by Zygmund [4]. But the question of characterizing the points of \((H, q)\)-summability of Fourier series for \( f \in L_1 \) remained open. In 1973 (see [5]) we characterized the points of \((H, 2)\)-summability of Fourier series for \( f \in L_1 \), i.e., we showed that for any summable function \( f \in L_1 \), \((H, 2)\)-summability of Fourier series holds at all points \( x \) such that

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{\nu=1}^{\lfloor zn \rfloor} \left\{ \frac{1}{\nu} \int_{\nu-1/n}^{\nu} |f(x + t) - f(x)| + |f(x - t) - f(x)| \, dt \right\} = 0, \quad \gamma = 2,
\]

where \([z]\) denotes the integral part of \( z \).

It was established in the same paper that (3) holds almost everywhere for any integrable function \( f(x) \) when \( \gamma > 1 \).

Later Novikov and Rodin [6] showed that \((H, q)\)-summability holds at points \( x \) where (3) is satisfied when \( 1 < \gamma = p < 2 \) and \( p + q = pq \).

The equality (3) can be written in the equivalent form

\[
\lim_{n \to \infty} \sum_{\nu=1}^{\lfloor zn \rfloor} \frac{1}{\nu} \int_{\nu-1/n}^{\nu} |f(x \pm t) - f(x)| \, dt = 0,
\]

or

\[
\lim_{n \to \infty} \sum_{\nu=1}^{\lfloor zn \rfloor} \frac{1}{\nu} \int_{\nu-1/n}^{\nu} |f(x + t) - f(x)| \, dt = 0.
\]
It is known [7, p. 488] that \((H, q)\)-summability implies \((H, q')\)-summability for \(0 < q' < q\); hence it suffices to investigate \((H, q)\)-summability for large \(q\).

In the present paper we obtain an estimate for the rate of convergence to zero of the quantities
\[
\frac{1}{n+1} \sum_{\nu=0}^{n} |f(x) - S_{\nu}(f; x)|^q, \quad q > 0,
\]
\[
\Gamma_n(f; x, p) = \left\{ \sum_{k} \left( \frac{n+1}{|k|+1} \int_{\Delta_k^{(n)}} |f(x+t) - f(x)| \, dt \right)^{1/p} \right\}^{1/p},
\]
from which follow all of the above-mentioned results.

where \(\Delta_k^{(n)} = \left[ \frac{\pi k}{n+1}, \frac{\pi (k+1)}{n+1} \right] \cap \Delta \), and \(\sum_{k} \), here and in what follows, signifies that the summation has the limits \(-(n+1) \leq k \leq n\).

We will give some properties of the \(\Gamma_n(f; x, p)\).

**Property 1** [5]. If \(f \in L_1\), then for any \(p > 1\) we have almost everywhere
\[
\lim_{n \to \infty} \Gamma_n(f; x, p) = 0.
\]

**Property 2.** If \(f(x)\) is a continuous function with modulus of continuity \(\omega(f; \delta)\), then
\[
\Gamma_n(f; x, p) \leq 2\pi C_p \omega(f; \frac{1}{\sqrt{n+1}}), \quad \left(C_p = \left\{ \sum_{k} \left( \frac{|k|+1}{n+1} \right)^{-p} \right\}^{1/p}, \quad p + q = pq \right).
\]

Indeed,
\[
\Gamma_n(f; x, p) \leq \left\{ \sum_{k} \left( \frac{n+1}{|k|+1} \int_{\Delta_k^{(n)}} \omega(f; t) \, dt \right)^{1/p} \right\}^{1/p} \leq \left\{ \sum_{k} \left( \frac{n+1}{|k|+1} \omega(f; n+1)^{-1/q} \right) \times \int_{\Delta_k^{(n)}} \omega(f; t) \, dt \right\}^{1/p} \leq \omega(f; (n+1)^{-1/q}) \left\{ \sum_{k} \left( \frac{n+1}{|k|+1} \times \left( \frac{|k|+1}{n+1} \frac{\pi}{n+1} + \frac{\pi}{n+1} \right)^{1/p} \right) \right\}^{1/p} \leq 2\pi C_p \omega(f; (n+1)^{-1/q}), \quad p > 1,
\]
as required.

**Property 3.** If \(f \in \text{Lip}_{q} \alpha, 0 \leq \alpha < 1/q\), then
\[
\Gamma_n(f; x, p) \leq MC_{q-\alpha/p} (n+1)^{-\alpha} = O((n+1)^{\alpha}).
\]

Indeed, for \(f \in \text{Lip}_{q} \alpha\) we have \(|f(x+t) - f(x)| \leq M |t|^{\alpha}; \) hence
\[
\Gamma_n(f; x, p) \leq \left\{ \sum_{k} \left( \frac{n+1}{|k|+1} \frac{M (|k|+1)^{\alpha} \frac{\pi}{n+1}}{n+1} \right)^{1/p} \right\}^{1/p} = M (n+1)^{-\alpha} \left\{ \sum_{k} \left( \frac{|k|+1}{|k|+1} \frac{\pi}{n+1} \right) \right\}^{1/p} \leq MC_{q-\alpha/p} (n+1)^{-\alpha}, \quad (1-\alpha) p > 1,
\]
as required.

**Property 4.** If \(f \in L_p \ (p > 1)\), then (7) holds at all Lebesgue points \(x\) of degree \(p\).

**Proof.** Suppose \(x\) is a Lebesgue point of degree \(p\). By Hölder's inequality,
\[
\int_{\Delta_k^{(n)}} |f(x+t) - f(x)| \, dt \leq \left( \frac{\pi}{n+1} \right)^{\alpha} \left\{ \int_{\Delta_k^{(n)}} |f(x+t) - f(x)^p \, dt \right\}^{1/p}, \quad p + q = pq.
\]

Therefore,