The study of singularly perturbed linear-quadratic control problems on a finite time interval has been carried out earlier by the method of boundary functions in a series of investigations (see survey [1]); in particular, the most complete solution is contained in the dissertation [2]. The averaging method has been applied for the construction of an asymptotic solution of the linear-quadratic control problems of standard systems in [3].

We consider the problem of the minimization of the functional

$$J[u] = \frac{1}{2} \int_0^T y'Fy(t, \varepsilon, \varepsilon) + \frac{1}{2} e^{-1} \left( y'F(t)g + u'R(t)u \right) dt$$

under the constraints

$$\begin{align*}
\dot{x} &= \varepsilon (a_{11}(t)x + a_{12}(t)z + b_1(t)u), \\
\dot{z} &= a_{21}(t)x + a_{22}(t)z + b_2(t)u, \\
x(0, \varepsilon) &= x^0, \\
z(0, \varepsilon) &= z^0,
\end{align*}$$

where $y = (x', z')'$, the prime denotes transposition, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $u \in \mathbb{R}$, $\varepsilon > 0$ is a small parameter.

In this paper, for the problem (1)-(3), with the aid of the method of the boundary functions and the averaging method, we construct an asymptotic expansion of the solution on an asymptotically large time interval.

In accordance with the meaning of the problem, the following condition must be satisfied.

I. The matrices $F$, $\mathcal{P}(t)$ are positive semidefinite, $R(t)$ is positive definite, where

$$\mathcal{P}(t) = \begin{pmatrix} P_1(t) & P_2(t) \\ P_2(t) & P_3(t) \end{pmatrix}, \quad R(t) = \begin{pmatrix} R_1(t) & R_2(t) \\ R_2(t) & R_3(t) \end{pmatrix}, \quad F = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix}.$$ 

For the simplicity of the computations we also assume that $F_3$ is positive definite.

We shall solve problem (1)-(3) with the aid of L. S. Pontryagin's maximum principle. We note that for problems of this type it is both a necessary and a sufficient condition. According to the maximum principle, the optimal control is equal to

$$u^*(t, \varepsilon) = R^{-1}(t)(b_1\psi_1 + b_2\psi_2)(t, \varepsilon).$$
where $\psi_1$ and $\tilde{\psi}_2 = \varepsilon \psi_2$ are the conjugate variables. Then the boundary value problem of the maximum principle takes the form

$$
\dot{x} = \varepsilon (a_{11}(t)x + a_{12}(t)z + S_1(t)\psi_1 + S_2(t)\tilde{\psi}_2),
\dot{\psi}_1 = \varepsilon (P_1(t)x + P_2(t)z - a_{12}(t)\psi_1 - a_{11}(t)\tilde{\psi}_2),
$$

$$
\varepsilon \dot{z} = a_{21}(t)x + a_{22}(t)z + S_3(t)\psi_1 + S_4(t)\tilde{\psi}_2,
\varepsilon \dot{\tilde{\psi}}_2 = P_2(t)x + P_3(t)z - a_{22}(t)\psi_1 - a_{21}(t)\tilde{\psi}_2,
$$

$$
x(0, \varepsilon) = x_0, \quad z(0, \varepsilon) = z_0, \quad \psi_1(e^{-1}, \varepsilon) = -F_1x(e^{-1}, \varepsilon) - F_2z(e^{-1}, \varepsilon),
$$

$$
\varepsilon \tilde{\psi}_2(e^{-1}, \varepsilon) = -F_2x(e^{-1}, \varepsilon) - F_3z(e^{-1}, \varepsilon),
$$

where $S_1 = b_1R^{-1}b_1', \quad S_2 = b_1R^{-1}b_2', \quad S_3 = b_2R^{-1}b_2'.

We assume that the following conditions hold.

II. The matrices $a_{ii}(t), \ b_i(t), \ i, \ j = 1, 2, P(t), \ R(t)$ are $2\pi$-periodic and $k + 2$ times differentiable.

III. For each fixed $t \in [0, \varepsilon^{-1}]$ the pair $(a_{22}(t), b_2(t))$ is completely controllable, while the pair $(C(t), a_{22}(t))$ is completely observable, where $C'C = P_3.$

We consider the matrix algebraic Riccati equation

$$
- Ka_{22}(t) - a'_{22}(t)K + KS_3(t)K - P_3(t) = 0.
$$

It is known [1] that, under the condition III, there exist $M(t)$ (a unique positive definite solution of the Riccati equation) and $N(t)$ (a unique negative definite solution) such that the eigenvalues of the matrices $\alpha(t) = a_{22}(t) - S_3(t)M(t)$ and $\beta(t) = a_{22}(t) - S_3(t)N(t)$ satisfy the inequalities

$$
Re(\lambda(\alpha(t))) \leq -\sigma < 0, \quad Re(\lambda(\beta(t))) \geq \sigma > 0, \quad t \in [0, \varepsilon^{-1}].
$$

Such a $\sigma$, not depending on $\varepsilon$, can be found by virtue of the $2\pi$-periodicity of the coefficients of the Riccati equation. In addition, there exists a matrix

$$
B(t) = \begin{pmatrix}
B_{11}(t) & B_{12}(t) \\
B_{21}(t) & B_{22}(t)
\end{pmatrix} = \begin{pmatrix}
E & E \\
-M(t) & -N(t)
\end{pmatrix},
$$

such that

$$
B^{-1}(t)G(t)B(t) = \begin{pmatrix}
\alpha(t) & 0 \\
0 & \beta(t)
\end{pmatrix},
$$

where

$$
G(t) = \begin{pmatrix}
a_{22}(t) & S_3(t) \\
P_3(t) & -a'_{22}(t)
\end{pmatrix},
$$

and $\det B_{11}(0) \neq 0, \ det B_{22}(e^{-1}) \neq 0$. We shall construct the asymptotic expansion of the solution of the boundary value problem (5), (6) in the form [4]

$$
Y(t, \varepsilon) = \bar{Y}(t, \varepsilon) + \Pi Y(\tau_0, \varepsilon) + QY(\tau_1, \varepsilon),
$$

where $\tau_0 = \varepsilon^{-1}, \tau_1 = (1 - \varepsilon^{-1})e^{-1}, \bar{Y}(t, \varepsilon)$ is the joint vector of the four variables $x, z, \psi_1, \tilde{\psi}_2,$

$$
\bar{Y}(t, \varepsilon) = \bar{Y}_0(t, \varepsilon) + \bar{Y}_1(t, \varepsilon) + ...,
\Pi Y(\tau_0, \varepsilon) = \Pi_0Y(\tau_0, \varepsilon) + e\Pi_1Y(\tau_0, \varepsilon) + ...,
QY(\tau_1, \varepsilon) = Q_0Y(\tau_1, \varepsilon) + eQ_1Y(\tau_1, \varepsilon) + ... .
$$

We introduce the expansion (9) into the equations (5) and into the supplementary conditions (6). After appropriate transformations we make equal the coefficients of the same powers of $\varepsilon$ in the left- and right-hand sides of the equations, depending on $\tau_0, \tau_1$ separately, and also the coefficients in the equations for the fast variables, depending on $t$. In the equations for the slow variables we make equal the coefficients, depending on $t$, of $\varepsilon^l$ in the left-hand side and of $\varepsilon^{l+1}$ in the right-hand side, $l = 0, 1, ...$. We obtain additional conditions by making equal the coefficients of the same powers in the left- and right-hand sides.